The amplitude equation
for weakly nonlinear reversible phase boundaries

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Abstract

This technical note is a complement to an earlier paper [Benzoni-Gavage & Rosini, Comput. Math. Appl. 2009], which aims at a deeper understanding of a basic model for propagating phase boundaries that was proved to admit surface waves [Benzoni-Gavage, Nonlinear Anal. 1998]. The amplitude equation governing the evolution of weakly nonlinear surface waves for that model is computed explicitly, and is eventually found to have enough symmetry properties for the associated Cauchy problem to be locally well-posed.

1 Introduction

The reader is assumed to be familiar with the weakly nonlinear theory developed in [3], which is very much inspired from the seminal work [4]. In particular, we keep the same notation as in [3, Section 2], and consider the liquid-vapor phase transition model described in [3, Paragraphs 3.1, 3.2]. Our goal is to make explicit the amplitude equation, which corresponds to [3, Equation (2.20)] in an abstract framework. We adopt slightly different conventions compared with [3, Paragraph 3.3]. Namely, the eigenmodes with nonzero real part for the state ahead of the phase transition are[1]:

$$\beta_1^- := \frac{a_\ell - i\, u_\ell\, \eta_0}{c_\ell^2 - u_\ell^2}\,, \quad \beta_1^+ := \frac{-a_\ell - i\, u_\ell\, \eta_0}{c_\ell^2 - u_\ell^2} = -\overline{\beta_1^-}\,, \quad a_\ell := -c_\ell \sqrt{(c_\ell^2 - u_\ell^2)\,|\check{\eta}|^2 - \eta_0^2}\,. \tag{1}$$

The eigenmodes with nonzero real part for the state behind the phase transition are:

$$\beta_2^- := \frac{-a_r + i\, u_r\, \eta_0}{c_r^2 - u_r^2}\,, \quad \beta_2^+ := \frac{a_r + i\, u_r\, \eta_0}{c_r^2 - u_r^2} = -\overline{\beta_2^-}\,, \quad a_r := c_r \sqrt{(c_r^2 - u_r^2)\,|\check{\eta}|^2 - \eta_0^2}\,. \tag{2}$$

[1] The main difference here with [3, Paragraph 3.3] is our sign convention for a_ℓ, the latter being denoted by α_ℓ in [3, Paragraph 3.3].

The remaining, purely imaginary, eigenmodes are:

$$\beta_3^+ = \cdots = \beta_{d+1}^+ := \frac{i\,\eta_0}{u_\ell}\,, \quad \beta_3^- = \cdots = \beta_{d+1}^- := -\frac{i\,\eta_0}{u_r}\,.$$

The corresponding right eigenvectors are:

$$R_1^\pm := \begin{pmatrix} r_1^\pm \\ 0 \end{pmatrix}, \quad r_1^- := \begin{pmatrix} -i\,\eta_0 + u_\ell\,\beta_1^- \\ i\,c_\ell^2\,\check{\eta} \\ -a_\ell \end{pmatrix}, \quad r_1^+ := \begin{pmatrix} i\,\eta_0 - u_\ell\,\beta_1^+ \\ -i\,c_\ell^2\,\check{\eta} \\ -a_\ell \end{pmatrix} = \overline{r_1^-}\,, \tag{3}$$

$$R_2^\pm := \begin{pmatrix} 0 \\ r_2^\pm \end{pmatrix}, \quad r_2^- := \begin{pmatrix} -i\,\eta_0 - u_r\,\beta_2^- \\ i\,c_r^2\,\check{\eta} \\ -a_r \end{pmatrix}, \quad r_2^+ := \begin{pmatrix} i\,\eta_0 + u_r\,\beta_2^+ \\ -i\,c_r^2\,\check{\eta} \\ -a_r \end{pmatrix} = \overline{r_2^-}\,, \tag{4}$$

$$\begin{aligned} &R_j^+ := \begin{pmatrix} r_j^+ \\ 0 \end{pmatrix}, \quad R_j^- := \begin{pmatrix} 0 \\ r_j^- \end{pmatrix}, \quad j = 3, \dots, d+1\,, \\ &r_j^+ := \begin{pmatrix} 0 \\ \eta_0\,\check{e}_{j-2} \\ u_\ell\,\check{\eta}\cdot\check{e}_{j-2} \end{pmatrix}, \quad r_j^- := \begin{pmatrix} 0 \\ \eta_0\,\check{e}_{j-2} \\ u_r\,\check{\eta}\cdot\check{e}_{j-2} \end{pmatrix}, \end{aligned} \tag{5}$$

with $\check{e}_1 := \check{\eta}$ and the $d-2$ vectors $\check{e}_2, \dots, \check{e}_{d-1} \in \mathbb{R}^{d-1}$ span $\check{\eta}^\perp$.

The left eigenvectors are:

$$\begin{aligned} L_1^\pm := \begin{pmatrix} \ell_1^\pm \\ 0 \end{pmatrix}, \quad \ell_1^- &:= \frac{c_\ell^2 - u_\ell^2}{2\,a_\ell\,(u_\ell\,a_\ell + i\,c_\ell^2\,\eta_0)} \begin{pmatrix} i\,\eta_0 - 2\,u_\ell\,\beta_1^+ \\ -i\,\check{\eta} \\ \beta_1^+ \end{pmatrix}, \\ \ell_1^+ &:= \frac{c_\ell^2 - u_\ell^2}{2\,a_\ell\,(u_\ell\,a_\ell - i\,c_\ell^2\,\eta_0)} \begin{pmatrix} -i\,\eta_0 + 2\,u_\ell\,\beta_1^- \\ i\,\check{\eta} \\ -\beta_1^- \end{pmatrix} = \overline{\ell_1^-}\,, \end{aligned}$$

$$\begin{aligned} L_2^\pm := \begin{pmatrix} 0 \\ \ell_2^\pm \end{pmatrix}, \quad \ell_2^- &:= \frac{c_r^2 - u_r^2}{2\,a_r\,(u_r\,a_r + i\,c_r^2\,\eta_0)} \begin{pmatrix} -i\,\eta_0 - 2\,u_r\,\beta_2^+ \\ i\,\check{\eta} \\ \beta_2^+ \end{pmatrix}, \\ \ell_2^+ &:= \frac{c_r^2 - u_r^2}{2\,a_r\,(u_r\,a_r - i\,c_r^2\,\eta_0)} \begin{pmatrix} i\,\eta_0 + 2\,u_r\,\beta_2^- \\ -i\,\check{\eta} \\ -\beta_2^- \end{pmatrix} = \overline{\ell_2^-}\,, \end{aligned}$$

$$\begin{aligned} &L_j^+ := \begin{pmatrix} \ell_j^+ \\ 0 \end{pmatrix}, \quad L_j^- := \begin{pmatrix} 0 \\ \ell_j^- \end{pmatrix}, \quad j = 3, \dots, d+1\,, \\ &\ell_3^+ := \frac{1}{\eta_0^2 + u_\ell^2\,|\check{\eta}|^2} \begin{pmatrix} u_\ell \\ -\eta_0/(u_\ell\,|\check{\eta}|^2)\,\check{\eta} \\ -1 \end{pmatrix}, \quad \ell_3^- := \frac{1}{\eta_0^2 + u_r^2\,|\check{\eta}|^2} \begin{pmatrix} -u_r \\ \eta_0/(u_r\,|\check{\eta}|^2)\,\check{\eta} \\ 1 \end{pmatrix}, \\ &\ell_j^+ := -\frac{1}{u_\ell\,\eta_0} \begin{pmatrix} 0 \\ \check{e}'_{j-2} \\ 0 \end{pmatrix}, \quad \ell_j^- := \frac{1}{u_r\,\eta_0} \begin{pmatrix} 0 \\ \check{e}'_{j-2} \\ 0 \end{pmatrix}, \quad j = 4, \dots, d+1\,, \end{aligned}$$

where the $d-2$ vectors $\check{e}'_2, \dots, \check{e}'_{d-1}$ belong to $\check{\eta}^\perp$ and form the dual basis of $\check{e}_2, \dots, \check{e}_{d-1}$. Unlike the choice in [3, page 1476], the left eigenvectors here satisfy the normalization property:

$$\left(L_i^\pm\right)^* \breve{\mathbb{A}}^d(\underline{v})\, R_j^\pm = \delta_{i,j}\,, \quad \left(L_i^\pm\right)^* \breve{\mathbb{A}}^d(\underline{v})\, R_j^\mp = 0\,.$$

After linearizing the jump conditions we are left with the matrices

$$H(\underline{v}) := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & u_\ell\, I_{d-1} & 0 & 0 & -u_r\, I_{d-1} & 0 \\ c_\ell^2 - u_\ell^2 & 0 & 2\,u_\ell & -(c_r^2 - u_r^2) & 0 & -2\,u_r \\ (c_\ell^2 - u_\ell^2)\, u_\ell & 0 & u_\ell^2 + \mu & -(c_r^2 - u_r^2)\, u_r & 0 & -u_r^2 - \mu \end{pmatrix}, \tag{6}$$

where

$$\mu := \frac{1}{2}\, u_\ell^2 + g(\rho_\ell) = \frac{1}{2}\, u_r^2 + g(\rho_r)\,, \tag{7}$$

and

$$J(\underline{v})\, \eta := \begin{pmatrix} [\rho]\, \eta_0 \\ [p]\, \check{\eta} \\ 0 \\ (\mu\, [\rho] - [p])\, \eta_0 \end{pmatrix}. \tag{8}$$

Here we have used the relation

$$\frac{1}{2}\, j\, [u] + [\rho\, f] = \mu\, [\rho] - [p]\,,$$

where μ is defined in (7), in order to simplify the last entry of $J(\underline{v})\, \eta$.

The Lopatinskii determinant is defined by

$$\Delta(\eta) := \det \begin{pmatrix} J(\underline{v})\, \eta & H(\underline{v})\, R_1^- & \dots & H(\underline{v})\, R_{d+1}^- \end{pmatrix}. \tag{9}$$

Each column vector in the above determinant is computed by using (8), (6) and (3), (4), (5) for the definition of the right eigenvectors. Then some simple manipulations on the rows and columns of the above determinant yield (see [1] for similar computations):

$$\Delta(\eta) = (-u_r\, \eta_0)^{d-2}\, u_r\, \det \begin{pmatrix} \check{e}_1 & \dots & \check{e}_{d-1} \end{pmatrix}\, \det \begin{pmatrix} [\rho]\, \eta_0 & a_\ell & a_r & |\check{\eta}|^2 \\ 0 & u_\ell\, a_\ell + i\, \eta_0\, c_\ell^2 & u_r\, a_r + i\, \eta_0\, c_r^2 & 2\, u_r\, |\check{\eta}|^2 \\ -[p]\, \eta_0 & i\, \eta_0\, u_\ell\, c_\ell^2 & i\, \eta_0\, u_r\, c_r^2 & u_r^2\, |\check{\eta}|^2 \\ [p] & -i\, u_\ell\, c_\ell^2 & -i\, u_r\, c_r^2 & \eta_0 \end{pmatrix}.$$

For future use, we introduce the quantity

$$\Upsilon := (-u_r\, \eta_0)^{d-2}\, u_r\, \det \begin{pmatrix} \check{e}_1 & \dots & \check{e}_{d-1} \end{pmatrix} \in \mathbb{R} \setminus \{0\}\,. \tag{10}$$

The expression of the Lopatinskii determinant then reduces to

$$\Delta(\eta) = -[\rho]\ [u]\, \Upsilon\, (\eta_0^2 + u_r^2\, |\check{\eta}|^2)\, \big(u_\ell\, u_r\, a_\ell\, a_r + c_\ell^2\, c_r^2\, \eta_0^2\big)\,. \tag{11}$$

We fix a root $\underline{\eta}$ of the Lopatinskii determinant (see [1] for the properties of such roots, in particular the location of $|\underline{\eta}_0|/|\underline{\check{\eta}}|$ with respect to various velocities associated with the phase transition). From now on, an underline refers to evaluation at the root $\underline{\eta}$ of the Lopatinskii determinant.

We now define a vector $\sigma \in \mathbb{C}^{d+2}$ by computing some of the minors of $\Delta(\underline{\eta})$. More precisely, the vectors $H(\underline{v})\,\underline{R}_1^-,\dots,H(\underline{v})\,\underline{R}_{d+1}^-$ are linearly independent and we can thus define a vector $\sigma \in \mathbb{C}^{d+2} \setminus \{0\}$ satisfying

$$\forall\, X \in \mathbb{C}^{d+2}\,,\quad \det \begin{pmatrix} X & H(\underline{v})\,\underline{R}_1^- & \dots & H(\underline{v})\,\underline{R}_{d+1}^- \end{pmatrix} = \sigma^*\,X\,.$$

This vector σ can be computed explicitly by performing some elementary manipulations on each minor of (9). We do not give the detailed calculations but rather give the expression of σ. We find

$$\sigma^* = \underline{\Upsilon}\, \begin{pmatrix} D_1 & \check{D}\,\underline{\check{\eta}}^T & D_{d+1} & D_{d+2} \end{pmatrix}\,, \tag{12}$$

where $\underline{\Upsilon}$ denotes the quantity Υ in (10) evaluated at the frequency $\underline{\eta}$, and[2]

$$\begin{aligned} D_1 + \mu\, D_{d+2} &:= -(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\big([u]\,c_\ell^2\,c_r^2\,\underline{\eta}_0 - i\,u_\ell\,u_r\,(c_r^2\,\underline{a}_\ell - c_\ell^2\,\underline{a}_r)\big)\,,\\ \check{D} &:= -[u]\,u_r\,\big(\underline{a}_\ell\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0) + \underline{a}_r\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\big)\,,\\ D_{d+1} &:= -i\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_r\,c_r^2\,\underline{a}_\ell - u_\ell\,c_\ell^2\,\underline{a}_r)\,,\\ D_{d+2} &:= [u]\,\underline{\eta}_0\,(\underline{a}_\ell\,\underline{a}_r + c_\ell^2\,c_r^2\,|\underline{\check{\eta}}|^2)\\ &\qquad + i\,\underline{\eta}_0^2\,(c_r^2\,\underline{a}_\ell - c_\ell^2\,\underline{a}_r) - i\,|\underline{\check{\eta}}|^2\,\big(u_r\,(u_\ell - 2\,u_r)\,c_r^2\,\underline{a}_\ell + u_\ell\,u_r\,c_\ell^2\,\underline{a}_r\big)\,. \end{aligned} \tag{13}$$

It remains to compute the coefficients γ_1, γ_2 satisfying:

$$J(\underline{v})\,\underline{\eta} + \gamma_1\,H(\underline{v})\,\underline{R}_1^- + \gamma_2\,H(\underline{v})\,\underline{R}_2^- = 0\,.$$

Observe that our convention differs from that in [3, page 1477]. We get:

$$\gamma_1 := \frac{[\rho]\,u_r\,\underline{\eta}_0}{u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0} = \frac{-i\,[\rho]\,c_r^2\,\underline{\eta}_0^2}{\underline{a}_\ell\,(u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)}\,,\quad \gamma_2 := \frac{-[\rho]\,u_\ell\,\underline{\eta}_0}{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0} = \frac{i\,[\rho]\,c_\ell^2\,\underline{\eta}_0^2}{\underline{a}_r\,(u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)}\,, \tag{14}$$

where the equalities follow from the relation $u_\ell\,u_r\,\underline{a}_\ell\,\underline{a}_r + c_\ell^2\,c_r^2\,\underline{\eta}_0^2 = 0$ that is satisfied by the root $\underline{\eta}$ of the Lopatinskii determinant. For notational convenience, we also set

$$\gamma_3 = \dots = \gamma_{d+1} := 0\,.$$

Following [3, Proposition 2.2], the evolution of a weakly nonlinear phase transition is governed by a scalar amplitude w obeying a nonlocal Burgers equation:

$$a_0(k)\,\partial_\tau \widehat{w}(\tau,k) + \int_{\mathbb{R}} a_1(k-k',k')\,\widehat{w}(\tau,k-k')\,\widehat{w}(\tau,k')\,\mathrm{d}k' = 0\,, \tag{15}$$

where a_0 and a_1 are given by Equations (2.24) and (2.25) in [3, page 1471]. With the present notation, this yields

$$a_0(k) = \begin{cases} \alpha_0/(i\,k) & \text{if } k > 0,\\ \overline{\alpha_0}/(i\,k) & \text{if } k < 0, \end{cases}$$

and α_0 is a complex number whose definition is recalled in Equation (17) below. The expression of the kernel a_1 is recalled and made explicit in Section 3 below.

[2] Since $\underline{a}_\ell$ is negative and $\underline{a}_r$ is positive, D_{d+1} is nonzero and we can thus check that σ is a nonzero vector.

2 Computation of the coefficient α_0

Proposition 1. *The coefficient α_0 in the expression of a_0 is given by*

$$\alpha_0 = -\frac{[\rho]\ [u]\,\underline{\Upsilon}}{\underline{\eta}_0}\,(\underline{\eta}_0^2 + u_r^2\,|\check{\underline{\eta}}|^2)\,\left\{u_\ell^2\,u_r^2\left(\frac{\underline{a}_\ell^2}{c_\ell^2}+\frac{\underline{a}_r^2}{c_r^2}\right)+2\,c_\ell^2\,c_r^2\,\underline{\eta}_0^2\right\}, \tag{16}$$

and it coincides with the derivative of the Lopatinskii determinant Δ with respect to η_0 at its root $\underline{\eta}$. In particular, α_0 is a nonzero real number.

Proof. We recall that the expression of α_0 is

$$\alpha_0 = \sigma^*\,[\tilde{f}_0(\underline{v})] + i\,\sigma^*\,H(\underline{v})\,\underline{R}_p^+\,\frac{(\underline{L}_p^+)^*\,\gamma_q\,\underline{R}_q^-}{\underline{\beta}_p^+ - \underline{\beta}_q^-}\,, \tag{17}$$

where we use Einstein's summation convention over repeated indices. We first observe that the Hermitian product $(\underline{L}_p^+)^*\,\underline{R}_q^-$, $q=1,2$, vanishes as soon as p is larger than 4. In the same way, the products $(\underline{L}_1^+)^*\,\underline{R}_2^-$, $(\underline{L}_3^+)^*\,\underline{R}_2^-$ and $(\underline{L}_2^+)^*\,\underline{R}_1^-$ vanish so the expression of α_0 reduces to the sum of four terms:

$$\alpha_0 = \sigma^*\,[\tilde{f}_0(\underline{v})] + i\,\sigma^*\,H(\underline{v})\,\underline{R}_2^+\,\frac{(\underline{L}_2^+)^*\,\gamma_2\,\underline{R}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-} + i\,\sigma^*\,H(\underline{v})\,\underline{R}_1^+\,\frac{(\underline{L}_1^+)^*\,\gamma_1\,\underline{R}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} + i\,\sigma^*\,H(\underline{v})\,\underline{R}_3^+\,\frac{(\underline{L}_3^+)^*\,\gamma_1\,\underline{R}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-}\,. \tag{18}$$

We now compute each of these four quantities separately. Using

$$[\tilde{f}_0(\underline{v})] = \frac{1}{\underline{\eta}_0}\,J(\underline{v})\,\underline{\eta} - \frac{1}{\underline{\eta}_0}\begin{pmatrix}0\\ [p]\,\check{\underline{\eta}}\\ 0\\ 0\end{pmatrix},$$

and the orthogonality relation $\sigma^*\,J(\underline{v})\,\underline{\eta}=0$, we get

$$\begin{aligned}\sigma^*\,[\tilde{f}_0(\underline{v})] &= -[\rho]\,\underline{\Upsilon}\,\frac{u_\ell\,u_r\,|\check{\underline{\eta}}|^2}{\underline{\eta}_0}\,\check{D}\\ &= [\rho]\,[u]\,\underline{\Upsilon}\left\{\frac{u_\ell\,u_r^2\,|\check{\underline{\eta}}|^2}{\underline{\eta}_0}\left(\underline{a}_\ell\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0) + \underline{a}_r\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\right)\right\}.\end{aligned} \tag{19}$$

We now turn to the second term on the right in (18). There holds

$$\frac{(\underline{L}_2^+)^*\,\gamma_2\,\underline{R}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-} = -\frac{c_r^2\,|\check{\underline{\eta}}|^2\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)}{2\,\underline{a}_r^2\,(\underline{\eta}_0^2 + u_r^2\,|\check{\underline{\eta}}|^2)}\,\gamma_2\,,$$

and we also compute

$$\begin{aligned}i\,\sigma^*\,H(\underline{v})\,\underline{R}_2^+ = i\,\sigma^*\,\overline{H(\underline{v})\,\underline{R}_2^-} &= 2\,\underline{\Upsilon}\,c_r^2\left\{\underline{\eta}_0\,(D_{d+1}+u_r\,D_{d+2}) - u_r\,|\check{\underline{\eta}}|^2\check{D}\right\}\\ &= 2\,[u]\,\underline{\Upsilon}\,c_r^2\,\underline{a}_r\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,(u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\,,\end{aligned} \tag{20}$$

where the latter relation is obtained by using the expressions (13). Recalling the expression (14) of γ_2, we obtain

$$i\,\sigma^*\,H(\underline{v})\,\underline{R}_2^+\,\frac{(\underline{L}_2^+)^*\,\gamma_2\,\underline{R}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-} = [\rho]\ [u]\,\underline{\Upsilon}\left\{i\,\frac{u_\ell\,u_r\,\underline{a}_\ell}{\underline{a}_r}\,c_r^2\,|\check{\underline{\eta}}|^2\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\right\}. \tag{21}$$

We now examine the third term on the right in (18). There holds:

$$\frac{(\underline{L}_1^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} = -\frac{c_\ell^2 \, |\check{\eta}|^2 \, (u_\ell \, \underline{a}_\ell - i \, c_\ell^2 \, \underline{\eta}_0)}{2 \, \underline{a}_\ell^2 \, (\underline{\eta}_0^2 + u_\ell^2 \, |\check{\eta}|^2)} \, \gamma_1 \, ,$$

and we also compute

$$\begin{aligned} i \, \sigma^* \, H(\underline{v}) \, \underline{R}_1^+ = i \, \sigma^* \, \overline{H(\underline{v}) \, \underline{R}_1^-} &= 2 \, \underline{\Upsilon} \, c_\ell^2 \, \Big\{ u_\ell \, |\check{\eta}|^2 \check{D} - \underline{\eta}_0 \, (D_{d+1} + u_\ell \, D_{d+2}) \Big\} \\ &= -2 \, [u] \, \underline{\Upsilon} \, c_\ell^2 \, \underline{a}_\ell \, (\underline{\eta}_0^2 + u_r^2 \, |\check{\eta}|^2) \, (u_\ell \, \underline{a}_r - i \, c_r^2 \, \underline{\eta}_0) \, . \end{aligned} \tag{22}$$

Using the expression (14) of γ_1, we obtain

$$i \, \sigma^* \, H(\underline{v}) \, \underline{R}_1^+ \, \frac{(\underline{L}_1^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_1^+ - \underline{\beta}_1^-} = [\rho] \; [u] \, \underline{\Upsilon} \, \left\{ i \, \frac{u_\ell \, u_r \, \underline{a}_r}{\underline{a}_\ell} \, c_\ell^2 \, |\check{\eta}|^2 \, \frac{\underline{\eta}_0^2 + u_r^2 \, |\check{\eta}|^2}{\underline{\eta}_0^2 + u_\ell^2 \, |\check{\eta}|^2} \, (u_\ell \, \underline{a}_\ell - i \, c_\ell^2 \, \underline{\eta}_0) \right\} . \tag{23}$$

It remains to compute the last term on the right in (18) and to add the four expressions. First we compute

$$\frac{(\underline{L}_3^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-} = -\frac{c_\ell^2}{\underline{\eta}_0^2 + u_\ell^2 \, |\check{\eta}|^2} \, \gamma_1 \, ,$$

and we also compute

$$\sigma^* \, H(\underline{v}) \, \underline{R}_3^+ = -[u] \, \underline{\Upsilon} \, u_\ell \, |\check{\eta}|^2 \, (2 \, D_{d+1} + (u_\ell + u_r) \, D_{d+2}) \, , \tag{24}$$

where we have used the relation (which amounts to $\sigma^* \, H(\underline{v}) \, \underline{R}_3^- = 0$):

$$D_1 + \underline{\eta}_0 \, \check{D} + 2 \, u_r \, D_{d+1} + (\mu + u_r^2) \, D_{d+2} = 0 \, .$$

The expression (24) can be factorized by using the definitions (13) of D_{d+1}, D_{d+2}, and we obtain

$$\sigma^* \, H(\underline{v}) \, \underline{R}_3^+ = -\frac{[u]^2 \, \underline{\Upsilon} \, u_\ell \, |\check{\eta}|^2}{\underline{\eta}_0} \, (\underline{\eta}_0^2 - u_\ell \, u_r \, |\check{\eta}|^2) \, \big(\underline{a}_r \, (u_r \, \underline{a}_\ell - i \, c_\ell^2 \, \underline{\eta}_0) + \underline{a}_\ell \, (u_\ell \, \underline{a}_r - i \, c_r^2 \, \underline{\eta}_0) \big) \, . \tag{25}$$

Using (14), we derive the expression

$$i \, \sigma^* \, H(\underline{v}) \, \underline{R}_3^+ \, \frac{(\underline{L}_3^+)^* \gamma_1 \underline{R}_1^-}{\underline{\beta}_3^+ - \underline{\beta}_1^-} = [\rho] \; [u] \, \underline{\Upsilon} \, \left\{ i \, [u] \, u_\ell \, c_\ell^2 \, |\check{\eta}|^2 \, \frac{\underline{\eta}_0^2 - u_\ell \, u_r \, |\check{\eta}|^2}{\underline{\eta}_0^2 + u_\ell^2 \, |\check{\eta}|^2} \, (u_r \, \underline{a}_r - i \, c_r^2 \, \underline{\eta}_0) \right\} . \tag{26}$$

According to the decomposition (18), the coefficient α_0 is the sum of the four quantities in (19), (21), (23) and (26). Factorizing $[\rho] \; [u] \, \underline{\Upsilon}$ in each term, we first observe that the imaginary part of the sum equals zero. We can thus simplify α_0 by retaining only the real part of each term. This leads to the expression

$$\begin{aligned} \frac{\underline{\eta}_0 \, \alpha_0}{[\rho] \; [u] \, \underline{\Upsilon}} = &-c_\ell^2 \, c_r^2 \, \underline{\eta}_0^2 \, (u_r^2 + u_\ell \, u_r) \, |\check{\eta}|^2 + u_\ell \, u_r \, \underline{a}_\ell \, \frac{c_r^2 \, \underline{\eta}_0^2}{\underline{a}_r} \, c_r^2 \, |\check{\eta}|^2 \\ &+ u_\ell \, u_r \, \underline{a}_r \, \frac{c_\ell^2 \, \underline{\eta}_0^2}{\underline{a}_\ell} \, c_\ell^2 \, |\check{\eta}|^2 \, \frac{\underline{\eta}_0^2 + u_r^2 \, |\check{\eta}|^2}{\underline{\eta}_0^2 + u_\ell^2 \, |\check{\eta}|^2} + (u_\ell \, u_r - u_\ell^2) \, |\check{\eta}|^2 \, c_\ell^2 \, c_r^2 \, \underline{\eta}_0^2 \, \frac{\underline{\eta}_0^2 - u_\ell \, u_r \, |\check{\eta}|^2}{\underline{\eta}_0^2 + u_\ell^2 \, |\check{\eta}|^2} \, . \end{aligned}$$

At this stage, some elementary manipulations lead to the expression (16) of α_0.

The link between α_0 and the partial derivative $\partial_{\eta_0}\Delta(\underline{\eta})$ comes from the relation

$$\underline{\eta}_0\,\frac{\partial}{\partial\eta_0}(u_\ell\,u_r\,a_\ell\,a_r+c_\ell^2\,c_r^2\,\eta_0^2)\Big|_{\underline{\eta}}=u_\ell^2\,u_r^2\,\left(\frac{\underline{a}_\ell^2}{c_\ell^2}+\frac{\underline{a}_r^2}{c_r^2}\right)+2\,c_\ell^2\,c_r^2\,\underline{\eta}_0^2\,,$$

which is obtained by differentiating (11) and the expressions (1), (2) with respect to η_0 and then evaluating at $\underline{\eta}$. $\square$

Since the coefficient α_0 is real, we obtain $a_0(k)=\alpha_0/(i\,k)$ for all $k\neq 0$. In particular, the amplitude equation (15) reduces to

$$\partial_\tau\widehat{w}(\tau,k)+\frac{i\,k}{\alpha_0}\int_{\mathbb{R}}a_1(k-k',k')\,\widehat{w}(\tau,k-k')\,\widehat{w}(\tau,k')\,\mathrm{d}k'=0\,.$$

As far as smooth solutions are concerned, the Cauchy problem associated with this kind of nonlocal Burgers equation is known to be locally well-posed under rather simple algebraic conditions (see [2]). These conditions are invariant under multiplication by a nonzero real constant, so that it is sufficient to investigate whether they are satisfied by the slightly simpler kernel $4\,\pi\,a_1$. This is the purpose of the next section.

3 Computation of the quadratic kernel

We define the kernel $q(k,k'):=4\,\pi\,a_1(k,k')$. Following [3], we can decompose q as follows:

$$q(k,k')=\sum_{j=1}^{5}q_j(k,k')\,,$$

where the kernels $q_1,\dots,q_5$ are given by[3]

$$q_1(k,k'):=\sigma(k+k')^*\sum_{j=0}^{d-1}\underline{\eta}_j\left\{\mathrm{d}\tilde{f}^j(v_r)\cdot(\widehat{r}_+(k,0)+\widehat{r}_+(k',0))-\mathrm{d}\tilde{f}^j(v_\ell)\cdot(\widehat{r}_-(k,0)+\widehat{r}_-(k',0))\right\}\,,\tag{27}$$

$$q_2(k,k'):=-\sigma(k+k')^*\left\{\mathrm{d}^2\tilde{f}^d(v_r)\cdot(\widehat{r}_+(k,0),\widehat{r}_+(k',0))-\mathrm{d}^2\tilde{f}^d(v_\ell)\cdot(\widehat{r}_-(k,0),\widehat{r}_-(k',0))\right\}\,,\tag{28}$$

$$q_3(k,k'):=i\,(k+k')\int_0^{+\infty}L(k+k',z)\,\mathrm{d}\mathbb{A}(v,\underline{\eta})\cdot\widehat{r}(k,z)\cdot\widehat{r}(k',z)\,\mathrm{d}z\,,\tag{29}$$

$$q_4(k,k'):=\int_0^{+\infty}L(k+k',z)\,\frac{\partial}{\partial z}\left(\mathrm{d}\breve{\mathbb{A}}^d(v)\cdot\widehat{r}(k,z)\cdot\widehat{r}(k',z)\right)\,\mathrm{d}z\,,\tag{30}$$

$$q_5(k,k'):=-\int_0^{+\infty}L(k+k',z)\,\breve{\mathbb{A}}(v,\underline{\eta})\left(\frac{\partial\widehat{r}}{\partial z}(k,z)+\frac{\partial\widehat{r}}{\partial z}(k',z)\right)\,\mathrm{d}z\,.\tag{31}$$

We first examine the kernel q_1 and derive its expression for all values of (k,k').

Lemma 1. *Let us define the quantity*

$$Q:=2\,[\rho]\,[u]\,\underline{\Upsilon}\,(\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2)\,(\,\underline{\beta}_1^-+\,\underline{\beta}_2^-)\,i\,u_\ell\,u_r\,\underline{a}_\ell\,\underline{a}_r\,\frac{u_\ell\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r-i\,c_r^2\,\underline{\eta}_0}\,.\tag{32}$$

[3]We keep the notation of [3] for the functions $\widehat{r}_\pm,\widehat{r}$ and so on.

Then the kernel q_1 in (27) satisfies

$$q_1(k,k') = \begin{cases} 0 & \text{if } k>0 \text{ and } k'>0,\\ \overline{Q} & \text{if } k>0,\ k'<0 \text{ and } k+k'>0. \end{cases}$$

Proof. The function g^0 is known to be an entropy for the isothermal Euler equations with corresponding flux $(g^1,\dots,g^d)$. We thus have the relations

$$\forall\, j=0,\dots,d\,,\quad \mathrm{d}g^0(v)\,A^j(v) = \mathrm{d}g^j(v)\,,$$

where we use the convention $A^0(v)=I$ for all v. Using this relation between the Jacobian matrices, we get

$$\sum_{j=0}^{d-1} \underline{\eta}_j\,\mathrm{d}\tilde{f}^j(v_r)\cdot\underline{r}_2^- = \begin{pmatrix} \sum_{j=0}^{d-1}\underline{\eta}_j\,A^j(v_r)\,\underline{r}_2^- \\ \mathrm{d}g^0(v_r)\,\sum_{j=0}^{d-1}\underline{\eta}_j\,A^j(v_r)\,\underline{r}_2^- \end{pmatrix} = i\;\underline{\beta}_2^-\,\begin{pmatrix} A^d(v_r)\,\underline{r}_2^- \\ \mathrm{d}g^d(v_r)\cdot\underline{r}_2^- \end{pmatrix} = -i\;\underline{\beta}_2^-\,H(\underline{v})\,\underline{R}_2^-\,.$$

Similarly we have

$$\sum_{j=0}^{d-1}\underline{\eta}_j\,\mathrm{d}\tilde{f}^j(v_\ell)\cdot\underline{r}_1^- = -i\;\underline{\beta}_1^-\,H(\underline{v})\,\underline{R}_1^-\,.$$

For $k>0$ and $k'>0$, we thus get

$$\begin{aligned} q_1(k,k') &= 2\,\sigma^*\sum_{j=0}^{d-1}\gamma_2\,\underline{\eta}_j\,\mathrm{d}\tilde{f}^j(v_r)\cdot\underline{r}_2^- - 2\,\sigma^*\sum_{j=0}^{d-1}\gamma_1\,\underline{\eta}_j\,\mathrm{d}\tilde{f}^j(v_\ell)\cdot\underline{r}_1^- \\ &= -2\,i\,\gamma_2\;\underline{\beta}_2^-\,\sigma^*\,H(\underline{v})\,\underline{R}_2^- + 2\,i\,\gamma_1\;\underline{\beta}_1^-\,\sigma^*\,H(\underline{v})\,\underline{R}_1^- = 0\,, \end{aligned}$$

because σ is orthogonal to both $H(\underline{v})\,\underline{R}_1^-$ and $H(\underline{v})\,\underline{R}_2^-$.

Let us now consider the case $k>0$, $k'<0$ and $k+k'>0$. Using the same relations as above for the differentials $\mathrm{d}g^j$, we obtain

$$\begin{aligned} q_1(k,k') &= \sigma^*\sum_{j=0}^{d-1}\underline{\eta}_j\,\mathrm{d}\tilde{f}^j(v_r)\cdot(\gamma_2\,\underline{r}_2^- + \overline{\gamma_2}\,\underline{r}_2^+) - \sigma^*\sum_{j=0}^{d-1}\underline{\eta}_j\,\mathrm{d}\tilde{f}^j(v_\ell)\cdot(\gamma_1\,\underline{r}_1^- + \overline{\gamma_1}\,\underline{r}_1^+) \\ &= -i\,\overline{\gamma_2}\;\underline{\beta}_2^+\,\sigma^*\,H(\underline{v})\,\underline{R}_2^+ + i\,\overline{\gamma_1}\;\underline{\beta}_1^+\,\sigma^*\,H(\underline{v})\,\underline{R}_1^+\,. \end{aligned}$$

The Hermitian products $\sigma^*\,H(\underline{v})\,\underline{R}_2^+$ and $\sigma^*\,H(\underline{v})\,\underline{R}_1^+$ have already been computed in the proof of Proposition 1, see (20) and (22). We then obtain

$$\begin{aligned} q_1(k,k') = &-2\,[u]\,\Upsilon\,(\underline{\eta}_0^2 + u_r^2\,|\check{\underline{\eta}}|^2)\,c_r^2\,\underline{a}_r\,(u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\,\overline{\gamma_2}\;\underline{\beta}_2^+ \\ &-2\,[u]\,\Upsilon\,(\underline{\eta}_0^2 + u_r^2\,|\check{\underline{\eta}}|^2)\,c_\ell^2\,\underline{a}_\ell\,(u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\,\overline{\gamma_1}\;\underline{\beta}_1^+\,. \end{aligned}$$

We use the definition (14) to obtain

$$c_r^2\,\underline{a}_r\,(u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\,\overline{\gamma_2} = c_\ell^2\,\underline{a}_\ell\,(u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\,\overline{\gamma_1} = i\,[\rho]\,c_\ell^2\,c_r^2\,\underline{\eta}_0^2\,\frac{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0}\,,$$

and the claim follows using the relation $c_\ell^2\,c_r^2\,\underline{\eta}_0^2 = -u_\ell\,u_r\,\underline{a}_\ell\,\underline{a}_r$. $\square$

Deriving the expression of the kernels q_3, q_4, q_5 requires the expression of the row vector $L(k+k', z)$, which we derive right now.

Lemma 2. *For $k>0$, there holds*

$$L(k,z) = \Big(\frac{\omega_1}{\gamma_1} \exp(-k\ \underline{\beta}_1^+ z)\, \widetilde{\ell}_1 + \frac{\omega_3}{\gamma_1} \exp(-k\ \underline{\beta}_3^+ z)\, \widetilde{\ell}_3 \quad \frac{\omega_2}{\gamma_2} \exp(-k\ \underline{\beta}_2^+ z)\, \widetilde{\ell}_2 \Big)\,,$$

where we have set

$$\widetilde{\ell}_1 := \big(i\,\eta_0 - 2\,u_\ell\ \underline{\beta}_1^+ \quad -i\,\check{\eta}^T \quad \underline{\beta}_1^+ \big)\,, \qquad \widetilde{\ell}_3 := \big(-u_\ell^2\,|\underline{\check{\eta}}|^2 \quad \eta_0\,\check{\eta}^T \quad u_\ell\,|\underline{\check{\eta}}|^2 \big)\,,$$
$$\widetilde{\ell}_2 := \big(-i\,\eta_0 - 2\,u_r\ \underline{\beta}_2^+ \quad i\,\check{\eta}^T \quad \underline{\beta}_2^+ \big)\,,$$

and

$$\omega_1 := [\rho]\,[u]\,\underline{\Upsilon}\,\frac{\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2}{\underline{\eta}_0^2 + u_\ell^2\,|\underline{\check{\eta}}|^2}\, i\,u_\ell\,\underline{\eta}_0\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0),$$
$$\omega_3 := [\rho]\,[u]^2\,\underline{\Upsilon}\,\frac{\underline{\eta}_0^2 - u_\ell\,u_r\,|\underline{\check{\eta}}|^2}{\underline{\eta}_0^2 + u_\ell^2\,|\underline{\check{\eta}}|^2}\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0), \qquad \omega_2 := [\rho]\,[u]\,\underline{\Upsilon}\, i\,u_r\,\underline{\eta}_0\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0).$$

Proof. We first observe that σ is orthogonal to the vectors $H(\underline{v})\,\underline{R}_p^+$ for $p \geq 4$, so for $k>0$ the expression of $L(k,z)$ reduces to

$$L(k,z) = \sum_{p=1}^{3} \sigma^*\,H(\underline{v})\,\underline{R}_p^+ \exp(-k\ \underline{\beta}_p^+ z)\,(\underline{L}_p^+)^*\,.$$

The expression of the products $\sigma^*\,H(\underline{v})\,\underline{R}_p^+$, $p=1,2,3$ can be found in (20), (22), (25), and we use the definitions of the left eigenvectors $\underline{L}_p^+$ to derive the expressions given in Lemma 2. $\square$

We now examine the kernel q_5, which, as q_1 but unlike q_2, q_3, q_4, does not contain any term in $p''(\rho_{\ell,r})$.

Lemma 3. *With Q defined in (32), the kernel q_5 in (31) satisfies*

$$q_5(k,k') = \begin{cases} 0 & \text{if } k>0 \text{ and } k'>0, \\ \overline{Q}\,\dfrac{k'}{k} & \text{if } k>0,\ k'<0 \text{ and } k+k'>0. \end{cases}$$

Proof. For $k>0$ and $k'>0$, we compute

$$\breve{\mathbb{A}}(v,\underline{\eta}) \left(\frac{\partial \widehat{r}}{\partial z}(k,z) + \frac{\partial \widehat{r}}{\partial z}(k',z) \right) = \begin{pmatrix} i\,\gamma_1\,(\underline{\beta}_1^-)^2\,(k \exp(k\ \underline{\beta}_1^- z) + k' \exp(k'\ \underline{\beta}_1^- z))\,A^d(v_\ell)\,\underline{r}_1^- \\ i\,\gamma_2\,(\underline{\beta}_2^-)^2\,(k \exp(k\ \underline{\beta}_2^- z) + k' \exp(k'\ \underline{\beta}_2^- z))\,A^d(v_r)\,\underline{r}_2^- \end{pmatrix}.$$

Using the orthogonality properties $\widetilde{\ell}_1\,A^d(v_\ell)\,\underline{r}_1^- = \widetilde{\ell}_3\,A^d(v_\ell)\,\underline{r}_1^- = \widetilde{\ell}_3\,A^d(v_r)\,\underline{r}_2^- = 0$, we get $q_5(k,k')=0$ if $k>0$ and $k'>0$ because the integrand in (31) vanishes.

Let us now consider the case $k>0$, $k'<0$ and $k+k'>0$. From the previous argument, we still find that the term $L(k+k',z)\,\breve{\mathbb{A}}(v,\underline{\eta})\,\partial_z \widehat{r}(k,z)$ vanishes. We thus get

$$\begin{aligned} q_5(k,k') &= -\int_0^{+\infty} L(k+k',z)\,\breve{\mathbb{A}}(v,\underline{\eta})\,\frac{\partial \widehat{r}}{\partial z}(k',z)\,\mathrm{d}z \\ &= -i\,k' \int_0^{+\infty} L(k+k',z) \begin{pmatrix} \overline{\gamma_1}\,(\underline{\beta}_1^+)^2 \exp(k'\ \underline{\beta}_1^+ z)\,A^d(v_\ell)\,\underline{r}_1^+ \\ \overline{\gamma_2}\,(\underline{\beta}_2^+)^2 \exp(k'\ \underline{\beta}_2^+ z)\,A^d(v_r)\,\underline{r}_2^+ \end{pmatrix} \mathrm{d}z\,. \end{aligned}$$

We now use the expression of $L(k+k',z)$ in Lemma 2. The expression of $q_5(k,k')$ is simplified by recalling the orthogonality property $\widetilde{\ell}_3\, A^d(v_\ell)\, \underline{r}_1^+ = 0$ and we get

$$
\begin{aligned}
q_5(k,k') &= -\,i\,k' \int_0^{+\infty} \frac{\overline{\gamma_1}}{\gamma_1}\,\omega_1\,\widetilde{\ell}_1\,A^d(v_\ell)\,\underline{r}_1^+\,(\,\underline{\beta}_1^+)^2\,\exp(-k\,\underline{\beta}_1^+\,z)\,\mathrm{d}z \\
&\quad -\,i\,k' \int_0^{+\infty} \frac{\overline{\gamma_2}}{\gamma_2}\,\omega_2\,\widetilde{\ell}_2\,A^d(v_r)\,\underline{r}_2^+\,(\,\underline{\beta}_2^+)^2\,\exp(-k\,\underline{\beta}_2^+\,z)\,\mathrm{d}z\,, \\
&= -\,i\,\frac{k'}{k}\,\left\{\frac{\overline{\gamma_1}}{\gamma_1}\,\omega_1\,\widetilde{\ell}_1\,A^d(v_\ell)\,\underline{r}_1^+\;\underline{\beta}_1^+ + \frac{\overline{\gamma_2}}{\gamma_2}\,\omega_2\,\widetilde{\ell}_2\,A^d(v_r)\,\underline{r}_2^+\;\underline{\beta}_2^+\right\}\,.
\end{aligned}
$$

The conclusion of Lemma 3 then follows from the relations

$$
\begin{aligned}
&\frac{\overline{\gamma_1}}{\gamma_1} = -\frac{\overline{\gamma_2}}{\gamma_2} = -\frac{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0}\,, \\
&\omega_1\,\widetilde{\ell}_1\,A^d(v_\ell)\,\underline{r}_1^+ = -\omega_2\,\widetilde{\ell}_2\,A^d(v_r)\,\underline{r}_2^+ = 2\,[\rho]\,[u]\,\Upsilon\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,u_\ell\,u_r\,\underline{a}_\ell\,\underline{a}_r\,.
\end{aligned}
$$

□

We immediately get

Corollary 1. *With Q defined in (32), the kernels q_1, q_5 in (27), (31) satisfy*

$$
(q_1+q_5)(k,k') = \begin{cases} 0 & \text{if } k>0 \text{ and } k'>0, \\ \overline{Q}\,\left(1+\dfrac{k'}{k}\right) & \text{if } k>0,\ k'<0 \text{ and } k+k'>0. \end{cases}
$$

Our goal now is to derive an explicit expression for the kernels q_2, q_3, q_4 in (28), (29), (30). This is more intricate because these kernels are quadratic with respect to the vector $\widehat{r}(k,z)$ and there is more algebra involved to obtain a factorized expression in each region of the (k,k')-plane. We begin with some preliminary computations that will be useful in Propositions 2 and 3 below.

Lemma 4. *The coordinates (13) of the vector σ satisfy*

$$
\begin{aligned}
\gamma_1\,\check{D} &= -[\rho]\,[u]\,u_r\,\underline{\eta}_0\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\,, \\
\gamma_2\,\check{D} &= [\rho]\,[u]\,u_r\,\underline{\eta}_0\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\,, \\
\gamma_1\,D_{d+1} &= -[\rho]\,u_r\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\,, \\
\gamma_2\,D_{d+1} &= -[\rho]\,u_\ell\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_r\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,, \\
\gamma_1\,D_{d+2} &= [\rho]\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_r\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0) - [\rho]\,[u]\,u_r\,|\underline{\check{\eta}}|^2\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\,, \\
\gamma_2\,D_{d+2} &= [\rho]\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_\ell\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0) + [\rho]\,[u]\,u_r\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\,.
\end{aligned}
$$

Proof. From the definition (13), there holds

$$
\check{D} = [u]\,u_r\,\big(\underline{a}_r\,(u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0) + \underline{a}_\ell\,(u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\big)\,,
$$

and we then use the expression (14) of γ_1, γ_2 to compute $\gamma_1\,\check{D}$ and $\gamma_2\,\check{D}$.

The expressions of $\gamma_{1,2}\,D_{d+1}$ follow from the observation that D_{d+1} satisfies

$$
\begin{aligned}
\underline{\eta}_0\,D_{d+1} &= -(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_r\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\,(u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0) \\
&= (\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,(u_r\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,(u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\,.
\end{aligned}
$$

We then use again (14).

To compute the product $\gamma_1\, D_{d+2}$, we recall the relation (22) which we found in the proof of Proposition 1. It reads

$$\underline{\eta}_0\,(D_{d+1}+u_\ell\,D_{d+2})-u_\ell\,|\check{\underline{\eta}}|^2\check{D}=[u]\,\underline{a}_\ell\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,(u_\ell\,\underline{a}_r-i\,c_r^2\,\underline{\eta}_0)\,.$$

We multiply the latter relation by γ_1 and use the previous expressions of $\gamma_1\,\check{D}$ and $\gamma_1\,D_{d+1}$ to obtain that of $\gamma_1\,D_{d+2}$. The expression of $\gamma_2\,D_{d+2}$ is then easily deduced by using $\gamma_2/\gamma_1=i\,c_\ell^2\,\underline{\eta}_0^2/(u_r\,\underline{a}_r)$. $\square$

We now compute the kernels q_2,q_3,q_4 in the region $\{k>0,k'>0\}$.

Proposition 2. *Let us define the quantities*

$$\begin{aligned}
Q_\ell &:= [\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,(\underline{\eta}_0^2+u_\ell^2\,|\check{\underline{\eta}}|^2)\,\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)\,,\\
Q_r &:= [\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,\frac{\underline{a}_\ell}{\underline{a}_r}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)^2\,\gamma_2\,(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-)\,,\\
Q_\sharp &:= 2\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,(\underline{\eta}_0^2+u_\ell\,u_r\,|\check{\underline{\eta}}|^2)\,i\,c_\ell^2\,c_r^2\,\underline{\eta}_0\,\left(\frac{c_r^2\,\gamma_2}{\rho_r\,u_r}-\frac{c_\ell^2\,\gamma_1}{\rho_\ell\,u_\ell}\right)\,.
\end{aligned}\tag{33}$$

Then the kernels q_2,q_3,q_4 defined in (28), (29) and (30) satisfy

$$(q_2+q_3+q_4)(k,k')=\left(\frac{p''(\rho_\ell)}{2}+\frac{c_\ell^2}{\rho_\ell}\right)\,Q_\ell+\left(\frac{p''(\rho_r)}{2}+\frac{c_r^2}{\rho_r}\right)\,Q_r+Q_\sharp\,,$$

for all $k>0$ and $k'>0$.

Proof. We first recall the expressions of the second differentials that are involved in the kernels q_2,q_3,q_4, see [3, page 1480]:

$$\sum_{k=1}^{d-1}\check{\underline{\eta}}_k\,\mathrm{d}^2f^k(v_{\ell,r})\cdot(v,v)=p''(\rho_{\ell,r})\begin{pmatrix}0\\ \rho^2\,\check{\underline{\eta}}\\ 0\end{pmatrix}+\frac{2}{\rho_{\ell,r}}\begin{pmatrix}0\\ \check{\underline{\eta}}\cdot\check{j}\,\check{j}\\ \check{\underline{\eta}}\cdot\check{j}\,(j_d-u_{\ell,r}\,\rho)\end{pmatrix}\,,\tag{34}$$

$$\begin{aligned}
\mathrm{d}^2\tilde{f}^d(v_{\ell,r})\cdot(v,v)=\begin{pmatrix}\mathrm{d}^2f^d(v_{\ell,r})\cdot(v,v)\\ \mathrm{d}^2g^d(v_{\ell,r})\cdot(v,v)\end{pmatrix}&=p''(\rho_{\ell,r})\begin{pmatrix}0\\0\\ \rho^2\\ u_{\ell,r}\,\rho^2\end{pmatrix}+\frac{2}{\rho_{\ell,r}}\begin{pmatrix}0\\ (j_d-u_{\ell,r}\,\rho)\,\check{j}\\ (j_d-u_{\ell,r}\,\rho)^2\\ 0\end{pmatrix}\\
&+\frac{1}{\rho_{\ell,r}}\begin{pmatrix}0\\0\\0\\ 3\,u_{\ell,r}\,(j_d-u_{\ell,r}\,\rho)^2-u_{\ell,r}\,c_{\ell,r}^2\,\rho^2+2\,c_{\ell,r}^2\,\rho\,j_d+u_{\ell,r}\,\check{j}\cdot\check{j}\end{pmatrix}\,.
\end{aligned}\tag{35}$$

The proof of Proposition 2 splits in several steps. We assume from now on that k and k' are both positive and we wish to compute the expression of the kernel $q_2+q_3+q_4$.

• Step 1: computation of the $p''(\rho_\ell)$ factor. In this first step, we collect all the terms that involve $p''(\rho_\ell)$ in $q_2+q_3+q_4$. The contribution of the kernel q_2 equals

$$\begin{aligned}
&\underline{\Upsilon}\,(D_{d+1}+u_\ell\,D_{d+2})\,\gamma_1^2\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\\
&\qquad=-[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\left\{(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,i\,c_r^2\,\underline{\eta}_0+(u_r\,\underline{a}_r-i\,c_r^2\,\underline{\eta}_0)\,u_\ell\,u_r\,|\check{\underline{\eta}}|^2\right\},
\end{aligned}\tag{36}$$

where we have used Lemma 4 to compute the product $(D_{d+1}+u_\ell\, D_{d+2})\,\gamma_1$.

The contribution of the kernel q_3 equals

$$\begin{aligned}
& i\,(k+k')\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,(-i\,\check{\underline{\eta}}^T)\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,\gamma_1^2\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\,\check{\underline{\eta}}\,\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z \\
&+i\,(k+k')\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,(\underline{\eta}_0\,\check{\underline{\eta}}^T)\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,\gamma_1^2\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\,\check{\underline{\eta}}\,\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z \\
&=\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\left\{\omega_1\,\frac{|\check{\underline{\eta}}|^2}{\underline{\beta}_1^+-\underline{\beta}_1^-}+\omega_3\,|\check{\underline{\eta}}|^2\,\frac{i\,\underline{\eta}_0}{\underline{\beta}_3^+-\underline{\beta}_1^-}\right\}.
\end{aligned}$$

Similarly, the contribution of the kernel q_4 reads

$$\begin{aligned}
&-\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,\underline{\beta}_1^+\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,\gamma_1^2\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\,(k+k')\,\underline{\beta}_1^-\,\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z \\
&-\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,u_\ell\,|\check{\underline{\eta}}|^2\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,\gamma_1^2\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\,(k+k')\,\underline{\beta}_1^-\,\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z \\
&=-\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\left\{\omega_1\,\frac{\underline{\beta}_1^+\,\underline{\beta}_1^-}{\underline{\beta}_1^+-\underline{\beta}_1^-}+\omega_3\,|\check{\underline{\eta}}|^2\,\frac{u_\ell\,\underline{\beta}_1^-}{\underline{\beta}_3^+-\underline{\beta}_1^-}\right\}.
\end{aligned}$$

Adding the contributions of q_3 and q_4 gives the term

$$\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\left\{-\frac{1}{2\,\underline{a}_\ell}\left(\frac{\underline{a}_\ell^2}{c_\ell^2}+c_\ell^2\,|\check{\underline{\eta}}|^2\right)\omega_1\,+u_\ell\,|\check{\underline{\eta}}|^2\,\omega_3\right\}. \tag{37}$$

We now use the definitions of ω_1 and ω_3, see Lemma 2, and add the contributions in (36) and (37) in order to obtain the $p''(\rho_\ell)$ term in $q_2+q_3+q_4$. We first obtain that the sum of the right hand side of (36) and the expression in (37) equals

$$-[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\,\frac{\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2}{2\,c_\ell^2\,(\underline{\eta}_0^2+u_\ell^2\,|\check{\underline{\eta}}|^2)}\,(u_\ell\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)\left(c_r^2\,\underline{\eta}_0^2+\frac{u_r\,\underline{a}_r}{u_\ell\,\underline{a}_\ell}\,u_\ell^2\,c_\ell^2\,|\check{\underline{\eta}}|^2\right).$$

This last expression is simplified a little further by observing that we have

$$(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)\,(u_\ell\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)=\frac{-1}{c_\ell^2-u_\ell^2}\,(u_\ell^2\,\underline{a}_\ell^2+c_\ell^4\,\underline{\eta}_0^2)=-c_\ell^2\,(\underline{\eta}_0^2+u_\ell^2\,|\check{\underline{\eta}}|^2)\,,$$

and

$$c_r^2\,\underline{\eta}_0^2+\frac{u_r\,\underline{a}_r}{u_\ell\,\underline{a}_\ell}\,u_\ell^2\,c_\ell^2\,|\check{\underline{\eta}}|^2=u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\left(c_\ell^2\,|\check{\underline{\eta}}|^2-\frac{\underline{a}_\ell^2}{c_\ell^2}\right)=u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\,(\underline{\eta}_0^2+u_\ell^2\,|\check{\underline{\eta}}|^2)\,.$$

Eventually, we find that the sum of all the terms that involve $p''(\rho_\ell)$ factorizes as $Q_\ell/2$ where Q_ℓ is defined in (33).

• Step 2: computation of the $p''(\rho_r)$ factor. We follow the same strategy as in the first step and compute the contribution that involves $p''(\rho_r)$ in each kernel. The contribution of the kernel q_2 equals

$$-\underline{\Upsilon}\,(D_{d+1}+u_r\,D_{d+2})\,\gamma_2^2\,(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-)^2=[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_2\,(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-)^2\,\frac{u_\ell\,\underline{a}_\ell}{c_r^2}\left\{i\,\underline{\eta}_0\,u_r\,\underline{a}_r-u_r^2\,c_r^2\,|\check{\underline{\eta}}|^2\right\}, \tag{38}$$

where we have used Lemma 4 to simplify $(D_{d+1} + u_r\, D_{d+2})\, \gamma_2$. The contribution of the kernel q_3 equals

$$i\,(k+k') \int_0^{+\infty} \frac{\omega_2}{\gamma_2}\, (i\, \underline{\check{\eta}}^T)\, \mathrm{e}^{-(k+k')\, \underline{\beta}_2^+\, z}\, \gamma_2^2\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2\, \underline{\check{\eta}}\, \mathrm{e}^{(k+k')\, \underline{\beta}_2^-\, z}\, \mathrm{d}z$$
$$= -\gamma_2\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2\, \omega_2\, \frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-}\,,$$

and the contribution of the kernel q_4 reads

$$\int_0^{+\infty} \frac{\omega_2}{\gamma_2}\, \underline{\beta}_2^+\, \mathrm{e}^{-(k+k')\, \underline{\beta}_2^+\, z}\, \gamma_2^2\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2\, (k+k')\, \underline{\beta}_2^-\, \mathrm{e}^{(k+k')\, \underline{\beta}_2^-\, z}\, \mathrm{d}z$$
$$= \gamma_2\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2\, \omega_2\, \frac{\underline{\beta}_2^+\, \underline{\beta}_2^-}{\underline{\beta}_2^+ - \underline{\beta}_2^-}\,.$$

Adding the contributions of q_3 and q_4 gives the term

$$-\gamma_2\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2\, \frac{\omega_2}{2\, \underline{a}_r} \left(\frac{\underline{a}_r^2}{c_r^2} + c_r^2\, |\underline{\check{\eta}}|^2 \right)\,.$$

When we add the latter term with the expression in (38), we obtain

$$-[\rho]\, [u]\, \underline{\Upsilon}\, \gamma_2\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2\, (\underline{\eta}_0^2 + u_r^2\, |\underline{\check{\eta}}|^2)\, \frac{i\, u_r\, \underline{\eta}_0}{2\, \underline{a}_r}\, (u_\ell\, \underline{a}_\ell + i\, c_\ell^2\, \underline{\eta}_0)\,,$$

and this quantity is further simplified by using the relation

$$(i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)\, i\, \underline{\eta}_0\, (u_\ell\, \underline{a}_\ell + i\, c_\ell^2\, \underline{\eta}_0) = -u_\ell\, \underline{a}_\ell\, \frac{u_r^2\, \underline{a}_r^2 + c_r^4\, \underline{\eta}_0^2}{c_r^2\, (c_r^2 - u_r^2)} = -u_\ell\, \underline{a}_\ell\, (\underline{\eta}_0^2 + u_r^2\, |\underline{\check{\eta}}|^2)\,.$$

Eventually, we find that the sum of all the terms that involve $p''(\rho_r)$ factorizes as $Q_r/2$.

• Step 3: computation of the remaining terms. In order to prove Proposition 2, we can assume from now on, and without loss of generality that $p''(\rho_\ell) = p''(\rho_r) = 0$ in (34) and (35). With this simplification, we compute

$$\sum_{k=1}^{d-1} \underline{\check{\eta}}_k\, \mathrm{d}^2 f^k(v_\ell) \cdot (\underline{r}_1^-, \underline{r}_1^-) = -\frac{2\, i\, c_\ell^4\, |\underline{\check{\eta}}|^2}{\rho_\ell} \begin{pmatrix} 0 \\ -i\, \underline{\check{\eta}} \\ \underline{\beta}_1^- \end{pmatrix}\,, \tag{39}$$

$$\mathrm{d}^2 \tilde{f}^d(v_\ell) \cdot (\underline{r}_1^-, \underline{r}_1^-) = \frac{2\, c_\ell^2}{\rho_\ell} \begin{pmatrix} 0 \\ -i\, c_\ell^2\, \underline{\beta}_1^-\, \underline{\check{\eta}} \\ c_\ell^2\, |\underline{\check{\eta}}|^2 + (i\, \underline{\eta}_0 - u_\ell\, \underline{\beta}_1^-)^2 \\ i\, c_\ell^2\, \underline{\eta}_0\, \underline{\beta}_1^- + u_\ell\, (i\, \underline{\eta}_0 - u_\ell\, \underline{\beta}_1^-)^2 \end{pmatrix}\,, \tag{40}$$

$$\sum_{k=1}^{d-1} \underline{\check{\eta}}_k\, \mathrm{d}^2 f^k(v_r) \cdot (\underline{r}_2^-, \underline{r}_2^-) = \frac{2\, i\, c_r^4\, |\underline{\check{\eta}}|^2}{\rho_r} \begin{pmatrix} 0 \\ i\, \underline{\check{\eta}} \\ \underline{\beta}_2^- \end{pmatrix}\,, \tag{41}$$

$$\mathrm{d}^2 \tilde{f}^d(v_r) \cdot (\underline{r}_2^-, \underline{r}_2^-) = \frac{2\, c_r^2}{\rho_r} \begin{pmatrix} 0 \\ i\, c_r^2\, \underline{\beta}_2^-\, \underline{\check{\eta}} \\ c_r^2\, |\underline{\check{\eta}}|^2 + (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2 \\ -i\, c_r^2\, \underline{\eta}_0\, \underline{\beta}_2^- + u_r\, (i\, \underline{\eta}_0 + u_r\, \underline{\beta}_2^-)^2 \end{pmatrix}\,, \tag{42}$$

where in (40) and (42), we have used the relations

$$c_\ell^2\,(\underline{\beta}_1^-)^2 = c_\ell^2\,|\underline{\check{\eta}}|^2 + (i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)^2\,,\quad c_r^2\,(\underline{\beta}_2^-)^2 = c_r^2\,|\underline{\check{\eta}}|^2 + (i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,.$$

With these expressions, let us look first at the kernel q_2 in (28). Using (40), we compute

$$\begin{aligned}\sigma^*\,\mathrm{d}^2\tilde{f}^d(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^-)\,\gamma_1^2 = \frac{2\,c_\ell^2}{\rho_\ell}\,\underline{\Upsilon}\,\gamma_1\,\Big\{&i\,c_\ell^2\,\underline{\beta}_1^-\,(\underline{\eta}_0\,\gamma_1\,D_{d+2} - |\underline{\check{\eta}}|^2\,\gamma_1\,\check{D}) + c_\ell^2\,|\underline{\check{\eta}}|^2\,\gamma_1\,D_{d+1}\\ &+(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)^2\,\gamma_1\,(D_{d+1} + u_\ell\,D_{d+2})\Big\}\,,\end{aligned}$$

and Lemma 4 turns this expression into

$$\begin{aligned}&\sigma^*\,\mathrm{d}^2\tilde{f}^d(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^-)\,\gamma_1^2\\ &\quad= \frac{2\,c_\ell^2}{\rho_\ell}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_1\,\Big\{u_r\,\underline{a}_r\,(u_\ell\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,\underline{\beta}_1^- - u_r\,c_\ell^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\Big\}\\ &\qquad - \frac{2\,c_\ell^2}{\rho_\ell}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_1\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)^2\,\Big\{(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,i\,c_r^2\,\underline{\eta}_0 + u_\ell\,u_r\,|\underline{\check{\eta}}|^2\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\Big\}\,.\end{aligned}\tag{43}$$

Similarly, we derive the relation

$$\begin{aligned}&\sigma^*\,\mathrm{d}^2\tilde{f}^d(v_r)\cdot(\underline{r}_2^-,\underline{r}_2^-)\,\gamma_2^2\\ &\quad= -\frac{2\,c_r^2}{\rho_r}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_2\,\Big\{u_\ell\,\underline{a}_\ell\,(u_r\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\,\underline{\beta}_2^- + u_\ell\,c_r^2\,|\underline{\check{\eta}}|^2\,(u_r\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\Big\}\\ &\qquad + \frac{2\,c_r^2}{\rho_r}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,\Big\{(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,i\,c_\ell^2\,\underline{\eta}_0 + u_r^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\Big\}\,.\end{aligned}\tag{44}$$

The kernels q_3, q_4 both read as a sum of two contributions, one from the system ahead of the phase boundary, and one from the system behind the phase boundary. We compute each of these contributions separately in order to combine them with either (43) or (44). Using (39), the 'left' contribution of q_3 equals

$$\begin{aligned}&i\,(k+k')\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,\widetilde{\ell}_1\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,\gamma_1^2\,\frac{-2\,i\,c_\ell^4\,|\underline{\check{\eta}}|^2}{\rho_\ell}\begin{pmatrix}0\\-i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z\\ &+i\,(k+k')\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,\widetilde{\ell}_3\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,\gamma_1^2\,\frac{-2\,i\,c_\ell^4\,|\underline{\check{\eta}}|^2}{\rho_\ell}\begin{pmatrix}0\\-i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z\\ &=\frac{2\,c_\ell^4}{\rho_\ell}\,\gamma_1\,\omega_1\,\frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-}\,\widetilde{\ell}_1\begin{pmatrix}0\\-i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix} + \frac{2\,c_\ell^4}{\rho_\ell}\,\gamma_1\,\omega_3\,\frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_3^+ - \underline{\beta}_1^-}\,\widetilde{\ell}_3\begin{pmatrix}0\\-i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}\,,\end{aligned}\tag{45}$$

and, similarly (using now (41) rather than (39)), the 'right' contribution of q_3 equals

$$i\,(k+k')\int_0^{+\infty}\frac{\omega_2}{\gamma_2}\,\widetilde{\ell}_2\,\mathrm{e}^{-(k+k')\,\underline{\beta}_2^+\,z}\,\gamma_2^2\,\frac{2\,i\,c_r^4\,|\underline{\check{\eta}}|^2}{\rho_r}\begin{pmatrix}0\\ i\,\underline{\check{\eta}}\\ \underline{\beta}_2^-\end{pmatrix}\mathrm{e}^{(k+k')\,\underline{\beta}_2^-\,z}\,\mathrm{d}z$$
$$=-\frac{2\,c_r^4}{\rho_r}\,\gamma_2\,\omega_2\,\frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_2^+-\underline{\beta}_2^-}\,\widetilde{\ell}_2\begin{pmatrix}0\\ i\,\underline{\check{\eta}}\\ \underline{\beta}_2^-\end{pmatrix}. \quad (46)$$

The kernel q_4 is computed by first observing that the vectors $\mathrm{d}^2 f^d(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^-)$, and $\mathrm{d}^2 f^d(v_r)\cdot(\underline{r}_2^-,\underline{r}_2^-)$ are obtained by retaining only the three first coordinates in (40) and (42):

$$\mathrm{d}^2 f^d(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^-)=\frac{2\,c_\ell^4\,\underline{\beta}_1^-}{\rho_\ell}\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix},\quad \mathrm{d}^2 f^d(v_r)\cdot(\underline{r}_2^-,\underline{r}_2^-)=\frac{2\,c_r^4\,\underline{\beta}_2^-}{\rho_r}\begin{pmatrix}0\\ i\,\underline{\check{\eta}}\\ \underline{\beta}_2^-\end{pmatrix}.$$

Consequently, the 'left' contribution of q_4 equals

$$-\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,\widetilde{\ell}_1\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,\gamma_1^2\,\frac{2\,c_\ell^4\,\underline{\beta}_1^-}{\rho_\ell}\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}(k+k')\,\underline{\beta}_1^-\,\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z$$
$$-\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,\widetilde{\ell}_1\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,\gamma_1^2\,\frac{2\,c_\ell^4\,\underline{\beta}_1^-}{\rho_\ell}\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}(k+k')\,\underline{\beta}_1^-\,\mathrm{e}^{(k+k')\,\underline{\beta}_1^-\,z}\,\mathrm{d}z$$
$$=-\frac{2\,c_\ell^4}{\rho_\ell}\,\gamma_1\,\omega_1\,\frac{(\underline{\beta}_1^-)^2}{\underline{\beta}_1^+-\underline{\beta}_1^-}\,\widetilde{\ell}_1\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}-\frac{2\,c_\ell^4}{\rho_\ell}\,\gamma_1\,\omega_3\,\frac{(\underline{\beta}_1^-)^2}{\underline{\beta}_3^+-\underline{\beta}_1^-}\,\widetilde{\ell}_3\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}, \quad (47)$$

and the 'right' contribution of q_4 equals

$$\frac{2\,c_r^4}{\rho_r}\,\gamma_2\,\omega_2\,\frac{(\underline{\beta}_2^-)^2}{\underline{\beta}_2^+-\underline{\beta}_2^-}\,\widetilde{\ell}_2\begin{pmatrix}0\\ i\,\underline{\check{\eta}}\\ \underline{\beta}_2^-\end{pmatrix}. \quad (48)$$

The 'left' contribution of q_3+q_4 is obtained by adding the expressions in (45) and (47), which gives

$$-\frac{2\,c_\ell^4}{\rho_\ell}\,\gamma_1\,\omega_1\,\frac{(\underline{\beta}_1^-)^2-|\underline{\check{\eta}}|^2}{\underline{\beta}_1^+-\underline{\beta}_1^-}\,\widetilde{\ell}_1\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}-\frac{2\,c_\ell^4}{\rho_\ell}\,\gamma_1\,\omega_3\,\frac{(\underline{\beta}_1^-)^2-|\underline{\check{\eta}}|^2}{\underline{\beta}_3^+-\underline{\beta}_1^-}\,\widetilde{\ell}_3\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}$$
$$=-\frac{2\,c_\ell^2}{\rho_\ell}\,\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\left\{\omega_1\,\widetilde{\ell}_1\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}+\omega_3\,\widetilde{\ell}_3\begin{pmatrix}0\\ -i\,\underline{\check{\eta}}\\ \underline{\beta}_1^-\end{pmatrix}\right\}$$
$$=\frac{c_\ell^2}{\rho_\ell\,\underline{a}_\ell}\,\gamma_1\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)^2\left\{(\underline{\eta}_0^2+u_\ell^2\,|\underline{\check{\eta}}|^2-2\,c_\ell^2\,|\underline{\check{\eta}}|^2)\,\omega_1+2\,u_\ell\,\underline{a}_\ell\,|\underline{\check{\eta}}|^2\,\omega_3\right\}. \quad (49)$$

The 'right' contribution of $q_3 + q_4$ is obtained by adding the expressions in (46) and (48), which gives

$$
\begin{aligned}
\frac{2\,c_r^4}{\rho_r}\,\gamma_2\,\omega_2\,\frac{(\underline{\beta}_2^-)^2 - |\underline{\check{\eta}}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-}\,\widetilde{\ell}_2\,\begin{pmatrix} 0 \\ i\,\underline{\check{\eta}} \\ \underline{\beta}_2^- \end{pmatrix} &= \frac{2\,c_r^2}{\rho_r}\,\gamma_2\,\omega_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,\frac{\underline{\beta}_2^+\,\underline{\beta}_2^- - |\underline{\check{\eta}}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-} \\
&= \frac{c_r^2}{\rho_r\,\underline{a}_r}\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2 - 2\,c_r^2\,|\underline{\check{\eta}}|^2)\,\omega_2\,. \qquad (50)
\end{aligned}
$$

We can now compute the 'left' contribution of the full kernel $q_2 + q_3 + q_4$ by combining the expression in (43) with the one in (49). We use here the expressions of ω_1 and ω_3 given in Lemma 2. The sum of (43) and (49) reads

$$
\begin{aligned}
A_\ell :=& \frac{2\,c_\ell^2}{\rho_\ell}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_1\,\Big\{u_r\,\underline{a}_r\,(u_\ell\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,\underline{\beta}_1^- - u_r\,c_\ell^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\Big\} \\
&- \frac{2\,c_\ell^2}{\rho_\ell}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_1\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)^2\,\Big\{(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,i\,c_r^2\,\underline{\eta}_0 + u_\ell\,u_r\,|\underline{\check{\eta}}|^2\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\Big\} \\
&+ \frac{c_\ell^2}{\rho_\ell\,\underline{a}_\ell}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_1\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)^2\,\Big\{2\,u_\ell\,\underline{a}_\ell\,|\underline{\check{\eta}}|^2\,[u]\,\frac{\underline{\eta}_0^2 - u_\ell\,u_r\,|\underline{\check{\eta}}|^2}{\underline{\eta}_0^2 + u_\ell^2\,|\underline{\check{\eta}}|^2}\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0) \\
&+ (\underline{\eta}_0^2 + u_\ell^2\,|\underline{\check{\eta}}|^2 - 2\,c_\ell^2\,|\underline{\check{\eta}}|^2)\,i\,u_\ell\,\underline{\eta}_0\,(u_r\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0)\,\frac{\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2}{\underline{\eta}_0^2 + u_\ell^2\,|\underline{\check{\eta}}|^2}\Big\}\,.
\end{aligned}
$$

The three last rows in the definition of A_ℓ are factorized after a little bit of algebra, and we get

$$
\begin{aligned}
A_\ell =& \frac{2\,c_\ell^2}{\rho_\ell}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_1\,\Big\{u_r\,\underline{a}_r\,(u_\ell\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,\underline{\beta}_1^- - u_r\,c_\ell^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\Big\} \\
&- \frac{1}{\rho_\ell}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_1\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)^2\,u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\,(u_\ell\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,.
\end{aligned}
$$

We have thus shown that the 'left' contribution A_ℓ of $q_2 + q_3 + q_4$ reads

$$
A_\ell = \frac{2\,c_\ell^2}{\rho_\ell}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_1\,\Big\{u_r\,\underline{a}_r\,(u_\ell\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\,\underline{\beta}_1^- - u_r\,c_\ell^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\Big\} + \frac{c_\ell^2}{\rho_\ell}\,Q_\ell\,. \qquad (51)
$$

The 'right' contribution of $q_2 + q_3 + q_4$ by combining the expression in (44) (with a minus sign, recall the definition (28) of the kernel q_2) with the one in (50):

$$
\begin{aligned}
A_r :=& \frac{2\,c_r^2}{\rho_r}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_2\,\Big\{u_\ell\,\underline{a}_\ell\,(u_r\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\,\underline{\beta}_2^- + u_\ell\,c_r^2\,|\underline{\check{\eta}}|^2\,(u_r\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\Big\} \\
&- \frac{2\,c_r^2}{\rho_r}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,\Big\{(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,i\,c_\ell^2\,\underline{\eta}_0 + u_r^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_\ell - i\,c_\ell^2\,\underline{\eta}_0)\Big\} \\
&+ \frac{c_r^2}{\rho_r\,\underline{a}_r}\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2 - 2\,c_r^2\,|\underline{\check{\eta}}|^2)\,\omega_2\,.
\end{aligned}
$$

Once again, the last two rows are factorized after a few calculations that we skip, and we get

$$
\begin{aligned}
A_r =& \frac{2\,c_r^2}{\rho_r}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_2\,\Big\{u_\ell\,\underline{a}_\ell\,(u_r\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\,\underline{\beta}_2^- + u_\ell\,c_r^2\,|\underline{\check{\eta}}|^2\,(u_r\,\underline{a}_\ell + i\,c_\ell^2\,\underline{\eta}_0)\Big\} \\
&- \frac{1}{\rho_r}\,[\rho]\,[u]\,\underline{\Upsilon}\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-)^2\,u_\ell\,u_r\,\frac{\underline{a}_\ell}{\underline{a}_r}\,(u_r\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0)\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,.
\end{aligned}
$$

We have thus shown that the 'left' contribution A_ℓ of $q_2+q_3+q_4$ reads

$$A_r = \frac{2\,c_r^2}{\rho_r}\,[\rho]\,\underline{\Upsilon}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,\gamma_2\,\Big\{u_\ell\,\underline{a}_\ell\,(u_r\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0)\;\underline{\beta}_2^-+u_\ell\,c_r^2\,|\check{\underline{\eta}}|^2\,(u_r\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)\Big\}+\frac{c_r^2}{\rho_r}\,Q_r\,. \tag{52}$$

The final step of the proof consists in simplifying the remaining terms in A_ℓ and A_r. More specifically, the first factor in the expression (51) of A_ℓ can be simplied as follows:

$$\begin{aligned}
&u_\ell\,\Big\{u_r\,\underline{a}_r\,(u_\ell\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)\;\underline{\beta}_1^- - u_r\,c_\ell^2\,|\check{\underline{\eta}}|^2\,(u_\ell\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0)\Big\}\\
&\qquad = u_r\,\underline{a}_r\,(u_\ell\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)\,(u_\ell\;\underline{\beta}_1^- - i\,\underline{\eta}_0) - c_\ell^2\,\underline{\eta}_0^2\,(u_r\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0) - u_\ell\,u_r\,c_\ell^2\,|\check{\underline{\eta}}|^2\,(u_\ell\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0)\\
&\qquad = u_r\,\underline{a}_r\,c_\ell^2\,(\underline{\eta}_0^2+u_\ell^2\,|\check{\underline{\eta}}|^2) - c_\ell^2\,\underline{\eta}_0^2\,(u_r\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0) - u_\ell\,u_r\,c_\ell^2\,|\check{\underline{\eta}}|^2\,(u_\ell\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0)\\
&\qquad = -\,i\,c_\ell^2\,c_r^2\,\underline{\eta}_0\,(\underline{\eta}_0^2+u_\ell\,u_r\,|\check{\underline{\eta}}|^2)\,.
\end{aligned}$$

Similarly, the first factor in the expression (52) of A_r can be simplied by using

$$u_r\,\Big\{u_\ell\,\underline{a}_\ell\,(u_r\,\underline{a}_r+i\,c_r^2\,\underline{\eta}_0)\;\underline{\beta}_2^-+u_\ell\,c_r^2\,|\check{\underline{\eta}}|^2\,(u_r\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)\Big\} = i\,c_\ell^2\,c_r^2\,\underline{\eta}_0\,(\underline{\eta}_0^2+u_\ell\,u_r\,|\check{\underline{\eta}}|^2)\,.$$

Using these two last simplifications, we can add (51) and (52) and obtain

$$(q_2+q_3+q_4)(k,k') = \frac{c_\ell^2}{\rho_\ell}\,Q_\ell+\frac{c_r^2}{\rho_r}\,Q_r+Q_\sharp\,,$$

with $Q_\sharp$ defined in (33). This completes the proof of Proposition 2. ☐

We now compute the kernel $q_2+q_3+q_4$ in the domain $\{k>0, k'>0, k+k'>0\}$.

Proposition 3. *Let Q_ℓ and Q_r be defined in (33) and let us define*

$$Q_\flat := -2\,[\rho]\,[u]\,\underline{\Upsilon}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,u_\ell\,u_r\,|\check{\underline{\eta}}|^2\,\left\{\frac{c_\ell^4\,\underline{a}_r}{\rho_\ell\,\underline{a}_\ell}\,\overline{\gamma_1}\,(i\,\underline{\eta}_0-u_\ell\;\underline{\beta}_1^+)+\frac{c_r^4\,\underline{a}_\ell}{\rho_r\,\underline{a}_r}\,\overline{\gamma_2}\,(i\,\underline{\eta}_0+u_r\;\underline{\beta}_2^+)\right\}\,.$$

Then the kernels q_2,q_3,q_4 defined in (28), (29) and (30) satisfy

$$(q_2+q_3+q_4)(k,k') = \left\{\left(\frac{p''(\rho_\ell)}{2}-\frac{c_\ell^2}{\rho_\ell}\right)\overline{Q_\ell}+\left(\frac{p''(\rho_r)}{2}-\frac{c_r^2}{\rho_r}\right)\overline{Q_r}+Q_\flat\right\}\left(1+\frac{k'}{k}\right)\,,$$

for all (k,k') such that $k>0$, $k'<0$ and $k+k'>0$.

Proof. We split again the proof in several steps, as was done in the proof of Proposition 2.

- Step 1: computation of the $p''(\rho_\ell)$ factor. We collect again the contributions that involve $p''(\rho_\ell)$. The contribution of the kernel q_2 equals

$$\begin{aligned}
&\underline{\Upsilon}\,(D_{d+1}+u_\ell\,D_{d+2})\,|\gamma_1|^2\,(-i\,\underline{\eta}_0+u_\ell\;\underline{\beta}_1^-)\,(i\,\underline{\eta}_0-u_\ell\;\underline{\beta}_1^+)\\
&\qquad = -[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\;\underline{\beta}_1^-|^2\,\Big\{(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,i\,c_r^2\,\underline{\eta}_0+(u_r\,\underline{a}_r-i\,c_r^2\,\underline{\eta}_0)\,u_\ell\,u_r\,|\check{\underline{\eta}}|^2\Big\}\,,
\end{aligned}\tag{53}$$

where the product $(D_{d+1}+u_\ell\,D_{d+2})\,\gamma_1$ has already been computed when deriving (36).

The contribution of the kernel q_3 equals

$$
\begin{aligned}
& i\,(k+k')\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,(-i\,\underline{\check{\eta}}^T)\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,|\gamma_1|^2\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,\underline{\check{\eta}}\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z \\
& +i\,(k+k')\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,(\underline{\eta}_0\,\underline{\check{\eta}}^T)\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,|\gamma_1|^2\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,\underline{\check{\eta}}\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z \\
& =\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\left\{\left(1+\frac{k'}{k}\right)\frac{\omega_1\,|\underline{\check{\eta}}|^2}{\underline{\beta}_1^+-\underline{\beta}_1^-}+\frac{i\,\underline{\eta}_0\,(k+k')\,u_\ell\,|\underline{\check{\eta}}|^2\,\omega_3}{k\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)+k'\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^+)}\right\}.
\end{aligned}
$$

Similarly, the contribution of the kernel q_4 reads

$$
\begin{aligned}
& -\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,\underline{\beta}_1^+\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,|\gamma_1|^2\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,(k\,\underline{\beta}_1^-+k'\,\underline{\beta}_1^+)\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z \\
& -\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,u_\ell\,|\underline{\check{\eta}}|^2\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,|\gamma_1|^2\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,(k\,\underline{\beta}_1^-+k'\,\underline{\beta}_1^+)\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z \\
& =-\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\left\{\frac{\omega_1}{\underline{\beta}_1^+-\underline{\beta}_1^-}\left(\underline{\beta}_1^+\,\underline{\beta}_1^-+\frac{k'}{k}\,(\underline{\beta}_1^+)^2\right)+\frac{(k\,u_\ell\,\underline{\beta}_1^-+k'\,u_\ell\,\underline{\beta}_1^+)\,u_\ell\,|\underline{\check{\eta}}|^2\,\omega_3}{k\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-)+k'\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^+)}\right\}.
\end{aligned}
$$

Adding the contributions of q_3 and q_4 gives the term

$$
\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\left\{-\frac{1}{2\,\underline{a}_\ell}\left(\frac{\underline{a}_\ell^2}{c_\ell^2}+c_\ell^2\,|\underline{\check{\eta}}|^2\right)\omega_1+u_\ell\,|\underline{\check{\eta}}|^2\,\omega_3+\frac{k'}{k}\,\omega_1\,\frac{|\underline{\check{\eta}}|^2-(\underline{\beta}_1^+)^2}{\underline{\beta}_1^+-\underline{\beta}_1^-}\right\}. \tag{54}
$$

We now add the contributions in (53) and (54) in order to obtain the $p''(\rho_\ell)$ term in $q_2+q_3+q_4$. There is first a constant term that is independent of (k,k'), and this term is entirely similar to the one derived when adding (36) and (37) (see Step 1 in the proof of Proposition 2). Namely, the constant term in the sum of (53) and (54) equals

$$
\begin{aligned}
& -[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,\frac{\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2}{2\,c_\ell^2\,(\underline{\eta}_0^2+u_\ell^2\,|\underline{\check{\eta}}|^2)}\,(u_\ell\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)\left(c_r^2\,\underline{\eta}_0^2+\frac{u_r\,\underline{a}_r}{u_\ell\,\underline{a}_\ell}\,u_\ell^2\,c_\ell^2\,|\underline{\check{\eta}}|^2\right) \\
& \qquad =\frac{1}{2}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\,(-i\,\underline{\eta}_0+u_\ell\,\underline{\beta}_1^+)\,(\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2)\left(c_r^2\,\underline{\eta}_0^2+\frac{u_r\,\underline{a}_r}{u_\ell\,\underline{a}_\ell}\,u_\ell^2\,c_\ell^2\,|\underline{\check{\eta}}|^2\right)=\frac{1}{2}\,\overline{Q_\ell}\,.
\end{aligned}
$$

The last contribution in the $p''(\rho_\ell)$ term is the one that depends on (k,k') in (54), that is

$$
\begin{aligned}
& \frac{k'}{k}\,\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,\omega_1\,\frac{|\underline{\check{\eta}}|^2-(\underline{\beta}_1^+)^2}{\underline{\beta}_1^+-\underline{\beta}_1^-} \\
& \qquad =\frac{k'}{k}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\,|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2\,\frac{\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2}{\underline{\eta}_0^2+u_\ell^2\,|\underline{\check{\eta}}|^2}\,\frac{(-u_\ell\,u_r\,\underline{a}_r)}{c_\ell^2}\,(u_\ell\,\underline{a}_\ell-i\,c_\ell^2\,\underline{\eta}_0)\,\frac{(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^+)^2}{2\,c_\ell^2\,\underline{a}_\ell}\,(c_\ell^2-u_\ell^2) \\
& \qquad =\frac{k'}{2\,k}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\,(-i\,\underline{\eta}_0+u_\ell\,\underline{\beta}_1^+)\,(\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2)\,u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\,\frac{|i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^-|^2}{c_\ell^2}\,(c_\ell^2-u_\ell^2)=\frac{k'}{2\,k}\,\overline{Q_\ell}\,.
\end{aligned}
$$

We have thus shown that the $p''(\rho_\ell)$ term in $(q_2+q_3+q_4)(k,k')$ is as claimed in Proposition 3.

- Step 2: computation of the $p''(\rho_r)$ factor. The contribution of the kernel q_2 equals

$$
-\underline{\Upsilon}\,(D_{d+1}+u_r\,D_{d+2})\,|\gamma_2|^2\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2=[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_2}\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,\frac{u_\ell\,\underline{a}_\ell}{c_r^2}\left\{i\,\underline{\eta}_0\,u_r\,\underline{a}_r-u_r^2\,c_r^2\,|\underline{\check{\eta}}|^2\right\}. \tag{55}
$$

The contribution of the kernel q_3 equals

$$i\,(k+k')\int_0^{+\infty}\frac{\omega_2}{\gamma_2}\,(i\,\underline{\check{\eta}}^T)\,\mathrm{e}^{-(k+k')\,\underline{\beta}_2^+\,z}\,|\gamma_2|^2\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,\underline{\check{\eta}}\,\mathrm{e}^{k\,\underline{\beta}_2^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_2^+\,z}\,\mathrm{d}z$$
$$=-\overline{\gamma_2}\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,\omega_2\,\frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_2^+-\underline{\beta}_2^-}\,\left(1+\frac{k'}{k}\right)\,,$$

and the contribution of the kernel q_4 reads

$$\int_0^{+\infty}\frac{\omega_2}{\gamma_2}\,\underline{\beta}_2^+\,\mathrm{e}^{-(k+k')\,\underline{\beta}_2^+\,z}\,|\gamma_2|^2\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,(k\,\underline{\beta}_2^-+k'\,\underline{\beta}_2^+)\,\mathrm{e}^{k\,\underline{\beta}_2^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_2^+\,z}\,\mathrm{d}z$$
$$=\overline{\gamma_2}\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,\frac{\omega_2}{\underline{\beta}_2^+-\underline{\beta}_2^-}\,\left(\underline{\beta}_2^+\,\underline{\beta}_2^-+\frac{k'}{k}\,(\underline{\beta}_2^+)^2\right)\,.$$

Adding the contributions of q_3 and q_4 gives the term

$$\overline{\gamma_2}\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\left\{-\frac{\omega_2}{2\,\underline{a}_r}\,\left(\frac{\underline{a}_r^2}{c_r^2}+c_r^2\,|\underline{\check{\eta}}|^2\right)+\frac{k'}{k}\,\omega_2\,\frac{(\underline{\beta}_2^+)^2-|\underline{\check{\eta}}|^2}{\underline{\beta}_2^+-\underline{\beta}_2^-}\right\}\,. \tag{56}$$

When we add the latter term with the expression in (55), we obtain the constant term (independent of (k,k')):

$$-[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_2}\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,(\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2)\,\frac{i\,u_r\,\underline{\eta}_0}{2\,\underline{a}_r}\,(u_\ell\,\underline{a}_\ell+i\,c_\ell^2\,\underline{\eta}_0)=\frac{1}{2}\,\overline{Q_r}\,.$$

The only term that depends on (k,k') arises in (56) and equals

$$\frac{k'}{k}\,\overline{\gamma_2}\,|i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^-|^2\,\omega_2\,\frac{(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^+)^2}{2\,c_r^2\,\underline{a}_r}\,(c_r^2-u_r^2)=\frac{k'}{2\,k}\,\overline{\gamma_2}\,(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^+)^2\,\frac{\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2}{\underline{a}_r}\,\omega_2=\frac{k'}{2\,k}\,\overline{Q_r}\,.$$

The sum of all the terms that involve $p''(\rho_r)$ factorizes as claimed in Proposition 3.

• Step 3: computation of the remaining terms. In order to prove Proposition 3, we can assume from now on, and without loss of generality that $p''(\rho_\ell)=p''(\rho_r)=0$ in (34) and (35). With this simplification, we compute

$$\sum_{k=1}^{d-1}\underline{\check{\eta}}_k\,\mathrm{d}^2 f^k(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^+)=-i\,\frac{c_\ell^4\,|\underline{\check{\eta}}|^2}{\rho_\ell}\,\begin{pmatrix}0\\2\,i\,\underline{\check{\eta}}\\-(\underline{\beta}_1^++\underline{\beta}_1^-)\end{pmatrix}\,, \tag{57}$$

$$\mathrm{d}^2\tilde{f}^d(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^+)=\frac{c_\ell^4}{\rho_\ell}\,\begin{pmatrix}0\\i\,(\underline{\beta}_1^++\underline{\beta}_1^-)\,\underline{\check{\eta}}\\-2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\\-2\,u_\ell\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}\,, \tag{58}$$

$$\sum_{k=1}^{d-1}\underline{\check{\eta}}_k\,\mathrm{d}^2 f^k(v_r)\cdot(\underline{r}_2^-,\underline{r}_2^+)=-i\,\frac{c_r^4\,|\underline{\check{\eta}}|^2}{\rho_r}\,\begin{pmatrix}0\\2\,i\,\underline{\check{\eta}}\\\underline{\beta}_2^++\underline{\beta}_2^-\end{pmatrix}\,, \tag{59}$$

$$\mathrm{d}^2\tilde{f}^d(v_r)\cdot(\underline{r}_2^-,\underline{r}_2^+)=-\frac{c_r^4}{\rho_r}\,\begin{pmatrix}0\\i\,(\underline{\beta}_2^++\underline{\beta}_2^-)\,\underline{\check{\eta}}\\2\,\underline{\beta}_2^+\,\underline{\beta}_2^-\\2\,u_r\,\underline{\beta}_2^+\,\underline{\beta}_2^-\end{pmatrix}\,, \tag{60}$$

As was done in Step 3 of the proof of Proposition 2, we are going to split the computations between the 'left' and 'right' contributions. Let us first concentrate on all 'left' terms. The 'left' contribution of q_2 is computed from (58) and equals

$$\begin{aligned}\sigma^*\,\mathrm{d}^2\tilde{f}^d(v_\ell)\cdot(\underline{r}_1^-,\underline{r}_1^+)\,|\gamma_1|^2 &= \sigma^*\,\frac{c_\ell^4}{\rho_\ell}\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\\ -2\,u_\ell\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}|\gamma_1|^2\\ &= \frac{2\,c_\ell^4}{\rho_\ell\,(c_\ell^2-u_\ell^2)}\,\underline{\Upsilon}\,\overline{\gamma_1}\Big\{u_\ell\,\underline{\eta}_0\,|\underline{\check{\eta}}|^2\,\gamma_1\,\check{D}-(\underline{\eta}_0^2-c_\ell^2\,|\underline{\check{\eta}}|^2)\,\gamma_1\,(D_{d+1}+u_\ell\,D_{d+2})\Big\}\,.\end{aligned}$$

Using Lemma 4, we find that the 'left' contribution of q_2 is given by

$$\frac{2\,c_\ell^4}{\rho_\ell\,(c_\ell^2-u_\ell^2)}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\Big\{i\,c_r^2\,\underline{\eta}_0\,(\underline{\eta}_0^2-c_\ell^2\,|\underline{\check{\eta}}|^2)\,(\underline{\eta}_0^2+u_r^2\,|\underline{\check{\eta}}|^2)-u_\ell\,u_r\,c_\ell^2\,|\underline{\check{\eta}}|^4\,(u_r\,\underline{a}_r-i\,c_r^2\,\underline{\eta}_0)\Big\}\,. \tag{61}$$

Similarly, we use (60) to compute the 'right' contribution of q_2, which yields

$$\begin{aligned}-\,\sigma^*\,\mathrm{d}^2\tilde{f}^d(v_\ell)\cdot(\underline{r}_2^-,\underline{r}_2^+)\,|\gamma_2|^2 &= \frac{2\,c_r^4}{\rho_r\,(c_r^2-u_r^2)}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_2}\\ &\times\Big\{(i\,c_\ell^2\,\underline{\eta}_0^3+u_\ell\,\underline{a}_\ell\,u_r^2\,|\underline{\check{\eta}}|^2)\,(\underline{\eta}_0^2-c_r^2\,|\underline{\check{\eta}}|^2)-\underline{\eta}_0^2\,u_r^2\,|\underline{\check{\eta}}|^2\,(u_\ell\,\underline{a}_\ell-i\,c_\ell^2\,\underline{\eta}_0)\Big\}\,.\end{aligned} \tag{62}$$

Using (57), the 'left' contribution of q_3 equals

$$\begin{aligned}&i\,(k+k')\int_0^{+\infty}\frac{\omega_1}{\gamma_1}\,\widetilde{\ell}_1\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,\frac{-i\,c_\ell^4\,|\underline{\check{\eta}}|^2}{\rho_\ell}\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ -(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\end{pmatrix}|\gamma_1|^2\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z\\ &+i\,(k+k')\int_0^{+\infty}\frac{\omega_3}{\gamma_1}\,\widetilde{\ell}_3\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,\frac{-i\,c_\ell^4\,|\underline{\check{\eta}}|^2}{\rho_\ell}\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ -(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\end{pmatrix}|\gamma_1|^2\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z\\ &=\frac{c_\ell^4}{\rho_\ell}\,\overline{\gamma_1}\,\omega_1\,\widetilde{\ell}_1\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ -(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\end{pmatrix}\frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_1^+ - \underline{\beta}_1^-}\left(1+\frac{k'}{k}\right)\\ &\quad\underbrace{+\,\frac{c_\ell^4}{\rho_\ell}\,\overline{\gamma_1}\,\omega_3\,\widetilde{\ell}_1\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ -(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\end{pmatrix}\frac{(k+k')\,|\underline{\check{\eta}}|^2}{k\,(\,\underline{\beta}_3^+ - \underline{\beta}_1^-)+k'\,(\,\underline{\beta}_3^+ - \underline{\beta}_1^+)}}_{\diamond}\,,\end{aligned} \tag{63}$$

and, similarly (using now (59) rather than (57)), the 'right' contribution of q_3 equals

$$\begin{aligned}&i\,(k+k')\int_0^{+\infty}\frac{\omega_2}{\gamma_2}\,\widetilde{\ell}_2\,\mathrm{e}^{-(k+k')\,\underline{\beta}_2^+\,z}\,\frac{-i\,c_r^4\,|\underline{\check{\eta}}|^2}{\rho_r}\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ \underline{\beta}_2^+ + \underline{\beta}_2^-\end{pmatrix}|\gamma_2|^2\,\mathrm{e}^{k\,\underline{\beta}_2^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_2^+\,z}\,\mathrm{d}z\\ &\qquad=\frac{c_r^4}{\rho_r}\,\overline{\gamma_2}\,\omega_2\,\widetilde{\ell}_2\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ \underline{\beta}_2^+ + \underline{\beta}_2^-\end{pmatrix}\frac{|\underline{\check{\eta}}|^2}{\underline{\beta}_2^+ - \underline{\beta}_2^-}\left(1+\frac{k'}{k}\right)\,.\end{aligned} \tag{64}$$

The 'left' contribution of q_4 is computed by retaining the three first rows in (58):

$$
\begin{aligned}
&-\int_0^{+\infty} \frac{\omega_1}{\gamma_1}\,\widetilde{\ell}_1\,\mathrm{e}^{-(k+k')\,\underline{\beta}_1^+\,z}\,\frac{c_\ell^4}{\rho_\ell}\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}|\gamma_1|^2\,(k\,\underline{\beta}_1^- + k'\,\underline{\beta}_1^+)\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z\\
&-\int_0^{+\infty} \frac{\omega_3}{\gamma_1}\,\widetilde{\ell}_1\,\mathrm{e}^{-(k+k')\,\underline{\beta}_3^+\,z}\,\frac{c_\ell^4}{\rho_\ell}\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}|\gamma_1|^2\,(k\,\underline{\beta}_1^- + k'\,\underline{\beta}_1^+)\,\mathrm{e}^{k\,\underline{\beta}_1^-\,z}\,\mathrm{e}^{k'\,\underline{\beta}_1^+\,z}\,\mathrm{d}z\\
=&-\frac{c_\ell^4}{\rho_\ell}\,\overline{\gamma_1}\,\omega_1\,\widetilde{\ell}_1\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}\frac{1}{\underline{\beta}_1^+ - \underline{\beta}_1^-}\left(\underline{\beta}_1^- + \frac{k'}{k}\,\underline{\beta}_1^+\right)\\
&\underbrace{-\frac{c_\ell^4}{\rho_\ell}\,\overline{\gamma_1}\,\omega_3\,\widetilde{\ell}_1\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}\frac{k\,\underline{\beta}_1^- + k'\,\underline{\beta}_1^+}{k\,(\,\underline{\beta}_3^+ - \underline{\beta}_1^-) + k'\,(\,\underline{\beta}_3^+ - \underline{\beta}_1^+)}}_{\diamond}\,,
\end{aligned}
\tag{65}
$$

and the 'right' contribution of q_4 equals

$$
\begin{aligned}
&-\frac{c_r^4}{\rho_r}\,\overline{\gamma_2}\,\omega_2\,\widetilde{\ell}_2\begin{pmatrix}0\\ i\,(\,\underline{\beta}_2^+ + \underline{\beta}_2^-)\,\underline{\check{\eta}}\\ 2\,\underline{\beta}_2^+\,\underline{\beta}_2^-\end{pmatrix}\frac{1}{\underline{\beta}_2^+ - \underline{\beta}_2^-}\left(\underline{\beta}_2^- + \frac{k'}{k}\,\underline{\beta}_2^+\right)\\
&\qquad=\frac{c_r^4}{\rho_r}\,\overline{\gamma_2}\,\omega_2\,\widetilde{\ell}_2\begin{pmatrix}0\\ i(\,\underline{\beta}_2^+ + \underline{\beta}_2^-)\underline{\check{\eta}}\\ 2\,\underline{\beta}_2^+\,\underline{\beta}_2^-\end{pmatrix} - \frac{c_r^4}{\rho_r}\,\overline{\gamma_2}\,\omega_2\,\widetilde{\ell}_2\begin{pmatrix}0\\ i(\,\underline{\beta}_2^+ + \underline{\beta}_2^-)\underline{\check{\eta}}\\ 2\,\underline{\beta}_2^+\,\underline{\beta}_2^-\end{pmatrix}\frac{\underline{\beta}_2^+}{\underline{\beta}_2^+ - \underline{\beta}_2^-}\left(1+\frac{k'}{k}\right).
\end{aligned}
\tag{66}
$$

The 'left' contribution of $q_3 + q_4$ is computed as follows. We first observe that the difference between the diamond term in (63) and the diamond term in (65) reads

$$
\frac{c_\ell^2}{\rho_\ell}\,\overline{\gamma_1}\,\omega_3\,u_\ell\,|\underline{\check{\eta}}|^2\left\{c_\ell^2\,|\underline{\check{\eta}}|^2 - c_\ell^2\,\underline{\beta}_1^+\,\underline{\beta}_1^- - (i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^+)\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-)\right\} = \frac{2\,c_\ell^6}{\rho_\ell\,(c_\ell^2 - u_\ell^2)}\,u_\ell\,|\underline{\check{\eta}}|^4\,\overline{\gamma_1}\,\omega_3\,.
$$

Consequently the 'left' contribution of $q_3 + q_4$, which corresponds to the sum of (63) and (65), equals

$$
\begin{aligned}
&\frac{c_\ell^4\,\overline{\gamma_1}\,\omega_1}{\rho_\ell\,(\,\underline{\beta}_1^+ - \underline{\beta}_1^-)}\left\{|\underline{\check{\eta}}|^2\,\widetilde{\ell}_1\begin{pmatrix}0\\ 2\,i\,\underline{\check{\eta}}\\ -(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\end{pmatrix} - \underline{\beta}_1^+\,\widetilde{\ell}_1\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix}\right\}\left(1+\frac{k'}{k}\right)\\
&+\frac{c_\ell^4}{\rho_\ell}\,\overline{\gamma_1}\,\omega_1\,\widetilde{\ell}_1\begin{pmatrix}0\\ i\,(\,\underline{\beta}_1^+ + \underline{\beta}_1^-)\,\underline{\check{\eta}}\\ -2\,\underline{\beta}_1^+\,\underline{\beta}_1^-\end{pmatrix} + \frac{2\,c_\ell^6}{\rho_\ell\,(c_\ell^2 - u_\ell^2)}\,u_\ell\,|\underline{\check{\eta}}|^4\,\overline{\gamma_1}\,\omega_3\\
=&\frac{2\,c_\ell^4}{\rho_\ell\,(\,\underline{\beta}_1^+ - \underline{\beta}_1^-)}\,\overline{\gamma_1}\,\omega_1\,(\,\underline{\beta}_1^+\,\underline{\beta}_1^- - |\underline{\check{\eta}}|^2)\,((\,\underline{\beta}_1^+)^2 - |\underline{\check{\eta}}|^2)\left(1+\frac{k'}{k}\right)\\
&-\frac{2\,c_\ell^4}{\rho_\ell\,(c_\ell^2 - u_\ell^2)}\,\overline{\gamma_1}\,\omega_1\left\{(\,\underline{\eta}_0^2 - c_\ell^2\,|\underline{\check{\eta}}|^2)\,\underline{\beta}_1^+ + i\,u_\ell\,\underline{\eta}_0\,|\underline{\check{\eta}}|^2\right\} + \frac{2\,c_\ell^6}{\rho_\ell\,(c_\ell^2 - u_\ell^2)}\,u_\ell\,|\underline{\check{\eta}}|^4\,\overline{\gamma_1}\,\omega_3\,.
\end{aligned}
$$

After a little bit of algebra, the 'left' contribution of $q_3 + q_4$ is found to be equal to

$$-\frac{c_\ell^2}{\rho_\ell}\,[\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,\left(\frac{\underline{a}_\ell^2}{c_\ell^2}+c_\ell^2\,|\check{\underline{\eta}}|^2\right)\,\overline{\gamma_1}\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^+)\,\left(1+\frac{k'}{k}\right)$$
$$+\frac{2\,c_\ell^4}{\rho_\ell\,(c_\ell^2-u_\ell^2)}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_1}\,\Big\{-i\,c_r^2\,\underline{\eta}_0\,(\underline{\eta}_0^2-c_\ell^2\,|\check{\underline{\eta}}|^2)\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)+u_\ell\,u_r\,c_\ell^2\,|\check{\underline{\eta}}|^4\,(u_r\,\underline{a}_r-i\,c_r^2\,\underline{\eta}_0)\Big\}\,.$$

When combined with (61), we have thus shown that the 'left' contribution of the kernel $q_2+q_3+q_4$ reads

$$-\frac{c_\ell^2}{\rho_\ell}\,[\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,\frac{\underline{a}_r}{\underline{a}_\ell}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,\left(\frac{\underline{a}_\ell^2}{c_\ell^2}+c_\ell^2\,|\check{\underline{\eta}}|^2\right)\,\overline{\gamma_1}\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^+)\,\left(1+\frac{k'}{k}\right)$$
$$=-\left(\frac{c_\ell^2}{\rho_\ell}\,\overline{Q_\ell}+2\,[\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,|\check{\underline{\eta}}|^2\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,\frac{c_\ell^4\,\underline{a}_r}{\rho_\ell\,\underline{a}_\ell}\,\overline{\gamma_1}\,(i\,\underline{\eta}_0-u_\ell\,\underline{\beta}_1^+)\right)\,\left(1+\frac{k'}{k}\right)\,. \tag{67}$$

Let us now compute the 'right' contribution of the kernel q_3+q_4. We add the expressions in (64) and (66), which gives

$$\frac{2\,c_r^4}{\rho_r}\,\overline{\gamma_2}\,\omega_2\,\left\{(\,\underline{\beta}_2^+\,\underline{\beta}_2^-\,)\,\underline{\beta}_2^+-\frac{\underline{\beta}_2^++\underline{\beta}_2^-}{2}\,|\check{\underline{\eta}}|^2\right\}-\frac{2\,c_r^4}{\rho_r}\,\overline{\gamma_2}\,\omega_2\,((\,\underline{\beta}_2^+)^2-|\check{\underline{\eta}}|^2)\,\frac{\underline{\beta}_2^+\,\underline{\beta}_2^--|\check{\underline{\eta}}|^2}{\underline{\beta}_2^+-\underline{\beta}_2^-}\,\left(1+\frac{k'}{k}\right)\,,$$

or equivalently

$$\frac{2\,c_r^4}{\rho_r\,(c_r^2-u_r^2)}\,[\rho]\,[u]\,\underline{\Upsilon}\,\overline{\gamma_2}\,i\,u_r\,\underline{\eta}_0\,(u_\ell\,\underline{a}_\ell-i\,c_\ell^2\,\underline{\eta}_0)\,\Big\{(\underline{\eta}_0^2-c_r^2\,|\check{\underline{\eta}}|^2)\,\underline{\beta}_2^+-i\,u_r\,\underline{\eta}_0\,|\check{\underline{\eta}}|^2\Big\}$$
$$-\frac{c_r^4}{\rho_r}\,[\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,\frac{\underline{a}_\ell}{\underline{a}_r}\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,\left(\frac{\underline{a}_r^2}{c_r^2}+c_r^2\,|\check{\underline{\eta}}|^2\right)\,\overline{\gamma_2}\,(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^+)\,\left(1+\frac{k'}{k}\right)\,. \tag{68}$$

The first row in (68) is exactly the opposite of (62), that is of the 'right' contribution of q_2. In other words, we have found that the 'right' contribution of $q_2+q_3+q_4$ is given by the second row in (68), which reads

$$-\left(\frac{c_r^2}{\rho_r}\,\overline{Q_r}+2\,[\rho]\,[u]\,\underline{\Upsilon}\,u_\ell\,u_r\,|\check{\underline{\eta}}|^2\,(\underline{\eta}_0^2+u_r^2\,|\check{\underline{\eta}}|^2)\,\frac{c_r^4\,\underline{a}_\ell}{\rho_r\,\underline{a}_r}\,\overline{\gamma_2}\,(i\,\underline{\eta}_0+u_r\,\underline{\beta}_2^+)\right)\,\left(1+\frac{k'}{k}\right)\,.$$

The kernel $q_2+q_3+q_4$ is the sum of the latter quantity and the right hand side of (67). Collecting the terms, this completes the proof of Proposition 3. $\square$

Corollary 2. *With Q defined in (32), $Q_\ell, Q_r, Q_\sharp, Q_\flat$ defined in Propositions 2 and 3, the kernel q satisfies*

$$q(k,k')=\begin{cases}\left(\dfrac{p''(\rho_\ell)}{2}+\dfrac{c_\ell^2}{\rho_\ell}\right)\,Q_\ell+\left(\dfrac{p''(\rho_r)}{2}+\dfrac{c_r^2}{\rho_r}\right)\,Q_r+Q_\sharp & \text{if } k>0 \text{ and } k'>0,\\ \left\{\left(\dfrac{p''(\rho_\ell)}{2}-\dfrac{c_\ell^2}{\rho_\ell}\right)\,\overline{Q_\ell}+\left(\dfrac{p''(\rho_r)}{2}-\dfrac{c_r^2}{\rho_r}\right)\,\overline{Q_r}+Q_\flat+\overline{Q}\right\}\,\left(1+\dfrac{k'}{k}\right) & \text{if } k>0, k'<0, k+k'>0.\end{cases}$$

4 Conclusion

There remains a final simplification in order to achieve the final form of the kernel q. Our main result reads as follows.

Proposition 4. *With $Q_\ell, Q_r, Q_\sharp$ defined in Propositions 2, let us define*

$$Q_\natural := \left(\frac{p''(\rho_\ell)}{2} + \frac{c_\ell^2}{\rho_\ell} \right) Q_\ell + \left(\frac{p''(\rho_r)}{2} + \frac{c_r^2}{\rho_r} \right) Q_r + Q_\sharp \, .$$

Then the kernel $q = 4\,\pi\,a_1$ satisfies

$$q(k,k') = \begin{cases} Q_\natural & \text{if } k>0 \text{ and } k'>0, \\ \overline{Q_\natural} \left(1 + \dfrac{k'}{k} \right) & \text{if } k>0,\ k'<0 \text{ and } k+k'>0. \end{cases}$$

In particular, q satisfies Hunter's stability condition $q(1,0^+) = \overline{q(1,0^-)}$, and (15) is locally well-posed in $H^2(\mathbb{R})$.

Proof. In view of the expression of q given in Corollary 2, we only need to show the relation

$$\frac{c_\ell^2}{\rho_\ell} Q_\ell + \frac{c_r^2}{\rho_r} Q_r + \frac{1}{2} Q_\sharp = \frac{1}{2} \left(Q + \overline{Q_\flat} \right) ,$$

with $Q_\ell, Q_r, Q_\sharp$ given in Proposition 2, $Q_\flat$ in Proposition 3, and Q in Lemma 1. Simplifying by the common factor $[\rho]\,[u]\,\underline{\Upsilon}\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,u_\ell\,u_r$, we are led to showing that the following relation holds:

$$\begin{aligned} &\frac{\underline{a}_r}{\rho_\ell\,\underline{a}_\ell}\, c_\ell^2\,(\underline{\eta}_0^2 + u_\ell^2\,|\underline{\check{\eta}}|^2)\,\gamma_1\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-) + \frac{\underline{a}_\ell}{\rho_r\,\underline{a}_r}\, c_r^2\,(\underline{\eta}_0^2 + u_r^2\,|\underline{\check{\eta}}|^2)\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-) \\ &\qquad - \frac{1}{\mathbf{j}\,[u]\,\underline{\eta}_0}\,(\underline{\eta}_0^2 + u_\ell\,u_r\,|\underline{\check{\eta}}|^2)\,i\,\underline{a}_\ell\,\underline{a}_r\,(c_r^2\,\gamma_2 - c_\ell^2\,\gamma_1) = (\underline{\beta}_1^- + \underline{\beta}_2^-)\,i\,\underline{a}_\ell\,\underline{a}_r\,\frac{u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0} \\ &\qquad\qquad + \frac{\underline{a}_r}{\rho_\ell\,\underline{a}_\ell}\, c_\ell^4\,|\underline{\check{\eta}}|^2\,\gamma_1\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-) + \frac{\underline{a}_\ell}{\rho_r\,\underline{a}_r}\, c_r^4\,|\underline{\check{\eta}}|^2\,\gamma_2\,(i\,\underline{\eta}_0 + u_r\,\underline{\beta}_2^-) \, , \end{aligned} \tag{69}$$

where we use the notation $\mathbf{j} := \rho_\ell\,u_\ell = \rho_r\,u_r$ to denote the mass flux across the phase boundary.

The verification of (69) proceeds as follows. We first combine the first and third row in (69) by recalling the definitions of $\underline{a}_\ell, \underline{a}_r$, see (1) and (2). Thus verifying (69) amounts to showing

$$\begin{aligned} &(\underline{\beta}_1^- + \underline{\beta}_2^-)\,\frac{u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0} - i\,\frac{\gamma_1}{\rho_\ell}\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-) - i\,\frac{\gamma_2}{\rho_r}\,(i\,\underline{\eta}_0 + u_\ell\,\underline{\beta}_2^-) \\ &\qquad + \frac{1}{\mathbf{j}\,[u]\,\underline{\eta}_0}\,(\underline{\eta}_0^2 + u_\ell\,u_r\,|\underline{\check{\eta}}|^2)\,(c_r^2\,\gamma_2 - c_\ell^2\,\gamma_1) = 0 \, . \end{aligned} \tag{70}$$

Let us now define

$$\begin{aligned} B_\ell &:= \underline{\beta}_1^-\,\frac{u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0} - i\,\frac{\gamma_1}{\rho_\ell}\,(i\,\underline{\eta}_0 - u_\ell\,\underline{\beta}_1^-) - \frac{1}{\mathbf{j}\,[u]\,\underline{\eta}_0}\,(\underline{\eta}_0^2 + u_\ell\,u_r\,|\underline{\check{\eta}}|^2)\,c_\ell^2\,\gamma_1 \, , \\ B_r &:= \underline{\beta}_2^-\,\frac{u_\ell\,\underline{a}_r + i\,c_r^2\,\underline{\eta}_0}{u_\ell\,\underline{a}_r - i\,c_r^2\,\underline{\eta}_0} - i\,\frac{\gamma_2}{\rho_r}\,(i\,\underline{\eta}_0 + u_\ell\,\underline{\beta}_2^-) + \frac{1}{\mathbf{j}\,[u]\,\underline{\eta}_0}\,(\underline{\eta}_0^2 + u_\ell\,u_r\,|\underline{\check{\eta}}|^2)\,c_r^2\,\gamma_2 \, , \end{aligned}$$

so that (70), which is the relation we wish to prove, reads $B_\ell + B_r = 0$.

Using the relations

$$\frac{u_\ell\, \underline{a}_r + i\, c_r^2\, \underline{\eta}_0}{u_\ell\, \underline{a}_r - i\, c_r^2\, \underline{\eta}_0} = -\frac{u_r\, \underline{a}_\ell + i\, c_\ell^2\, \underline{\eta}_0}{u_r\, \underline{a}_\ell - i\, c_\ell^2\, \underline{\eta}_0}\,, \quad (u_r\, \underline{a}_\ell - i\, c_\ell^2\, \underline{\eta}_0)\, \gamma_1 = -\rho_\ell\, [u]\, \underline{\eta}_0\,,$$

we get

$$u_\ell\, (u_r\, \underline{a}_\ell - i\, c_\ell^2\, \underline{\eta}_0)\, B_\ell = -i\, \underline{\eta}_0\, (u_\ell\, \underline{a}_\ell + i\, c_\ell^2\, \underline{\eta}_0)\,. \tag{71}$$

Similarly, we compute

$$u_r\, (u_\ell\, \underline{a}_r - i\, c_r^2\, \underline{\eta}_0)\, B_r = -i\, \underline{\eta}_0\, (u_r\, \underline{a}_r + i\, c_r^2\, \underline{\eta}_0)\,. \tag{72}$$

Combining the relations (71) and (72), we get

$$\begin{aligned}
&u_\ell\, u_r\, (u_r\, \underline{a}_\ell - i\, c_\ell^2\, \underline{\eta}_0)\, (u_\ell\, \underline{a}_r - i\, c_r^2\, \underline{\eta}_0)\, (B_\ell + B_r) \\
&\quad = -i\, \underline{\eta}_0\, u_r\, (u_\ell\, \underline{a}_r - i\, c_r^2\, \underline{\eta}_0)\, (u_\ell\, \underline{a}_\ell + i\, c_\ell^2\, \underline{\eta}_0) - i\, \underline{\eta}_0\, u_\ell\, (u_r\, \underline{a}_\ell - i\, c_\ell^2\, \underline{\eta}_0)\, (u_r\, \underline{a}_r + i\, c_r^2\, \underline{\eta}_0) \\
&\quad = -i\, \underline{\eta}_0\, (u_\ell + u_r)\, (u_\ell\, u_r\, \underline{a}_\ell\, \underline{a}_r + c_\ell^2\, c_r^2\, \underline{\eta}_0^2) = 0\,.
\end{aligned}$$

This means that (70) holds, and consequently the expression of the kernel q is as claimed in Proposition 4. The verification of Hunter's stability condition $q(1, 0^+) = \overline{q(1, 0^-)}$ is then straightforward, and local well-posedness in $H^2(\mathbb{R})$ follows from the main result in [2]. $\square$

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